CO 456 Homework #2.

Due at the beginning of class on Thursday September 25. Include your name and ID # on your solutions.

Problem 2–1. (A game of your own: 3 points)

Independent Problem. *Do not discuss this problem with anyone except for course staff.* Let the digits of your student ID number be $\#_1\#_2\ldots\#_8$. Form a 2-player strategic game from them by entering the digits according to the following pattern:

$p1\backslash p2$	L	R
T	$\#_1, \#_5$	$\#_8, \#_3$
B	$\#_4, \#_6$	$\#_7, \#_2$

As we showed in class, put a star over each $u_i(a)$ such that a_i is a best response to a_{-i} for player i. What are the Nash equilibria of your game?

Problem 2–2. (Cost of anarchy in simple routing)

Independent Problem. *Do not discuss this problem with anyone except for course staff.* In class we mentioned the *routing game*: each member of the population has to choose one of k roads $R_1, \ldots, R_k$ to take. Each road has a *delay function* d_i and the payoff of a player using R_i depends only on the delay of road R_i in the current action profile:

$$u_{R_i}(f) = -d_i(f_{R_i}).$$

(Recall again that f in this case, which represents an action profile, consists of nonnegative numbers $f_{R_1}, \ldots, f_{R_k}$ that add up to 1; see the online notes for the definition of a pNE.)

Problem 2–2(a). (3 points)

Suppose that $k = 2$ and the delay functions are $d_1(x) = x^p, d_2(x) = 1$ where $p > 0$ is a fixed real number. Find all pseudo-Nash equilibria of this game.

Problem 2–2(b). (1 point)

For each pseudo-Nash equilibrium f^* that you found in part **(a)**, compute its *average delay*, which we denote by $\overline{d}$ and define by

$$\overline{d}(f) = \sum_{i=1}^{k} f_{R_i} d_i(f_{R_i}).$$

Problem 2–2(c). (2 points)

Determine the action profile f^{opt} of this game with minimum possible average delay. (Note, f^{opt} has nothing to do with (pseudo-)Nash equilibria.)

Problem 2–2(d). (2 points)

As $p \to \infty$, what is the average delay of the action profile f^{opt} you found in part **(c)**? Compare with the result of part **(b)** and comment.

Problem 2–3. (Cournot model with many firms, modified from Osborne 61.1)

Collaborative Problem. *You can work on this problem with up to 3 of the other students in this class, but you must write up your solutions independently (without copying) and you must list all of your collaborators.* In this problem we generalize a result from class and find the set of all Nash equilibria in the simple linear Cournot model for an *arbitrary* number n of players. Assume as before that 1) the inverse demand function is $P(Q) = \max\{\alpha - Q, 0\}$, 2) each firm i has cost function $C_i(q_i) = cq_i$ where c is a fixed constant, and 3) $0 < c < \alpha$ so positive profit is possible.

Problem 2–3(a). (2 points)

Show that any outcome q where the total output Q satisfies $Q > \alpha - c$ is not a Nash equilibrium.

Problem 2–3(b). (2 points)

Show that any outcome q where the total output Q satisfies $Q = \alpha - c$ is not a Nash equilibrium.

Problem 2–3(c). (4 points)

Suppose q is a Nash equilibrium with $Q < \alpha - c$. Show that $q_i = \alpha - Q - c$ for each player i. (Hint: draw $u_i(q'_i, q_{-i})$ as a function of q'_i, obtaining a diagram that somewhat resembles Figure 58.1 of Osborne; observe that there is a unique best response to q_{-i} and compute its value.)

Problem 2–3(d). (2 points)

Find the set of all Nash equilibria. (It is helpful to check that your answer matches the answer we got in class for the special case $n = 2$.) Verify that as $n \to \infty$ the price $P(Q)$ approaches c.

Problem 2–4. (Discretized Bertrand's model: Osborne 67.2)

Collaborative Problem. *You can work on this problem with up to 3 of the other students in this class, but you must write up your solutions independently (without copying) and you must list all of your collaborators.* Consider Bertrand's duopoly game where there are $n = 2$ firms, the cost functions are $C_i(q_i) = cq_i$ for $i = 1, 2$, and the demand function is $D(p) = \max\{0, \alpha - p\}$. Restrict each firm to choose an *integral* price (e.g., representing an integral number of cents). Assume that c is an integer and also assume that $\alpha > c + 1$ and $c > 0$ so that positive profit is possible.

Problem 2–4(a). (1 point)

Is (c, c) still a Nash equilibrium? Provide proof.

Problem 2–4(b). (3 points)

Are there any other Nash equilibria? Provide proof.

Problem 2–5. (Flatland Elections)

Fun Problem. *This problem is not for credit; don't hand in a solution.* Find all Nash equilibria of Hotelling's game (from lecture / Section 3.3 of Osborne) when the voters are not on a line, but instead they are uniformly distributed in a circle/square/sphere/Borg cube.