CO 456 Homework #6.

Due at the beginning of class on **Tuesday November 25**.

About the final project: Your presentation should have about 12 minutes of speaking, leaving about 4 minutes for questions and to set up the next talk. You can use the blackboard, overhead projector, or a computer. To expedite setup, we'll try for all electronic presentations to be done from Dave's laptop — you can either email your presentation (.pdf or .ppt) to dagpritc@math.uwaterloo.ca or bring a USB key. But if for some reason you need another computer, e.g. if you need software only available on a Mac, that's okay.

Tentative schedule — tell me ASAP if you have a conflict:

- Nov 18: Moniz, Muzzerall/Kalai, Byrne/Lopyrev, Denisenkov, Santos
- Nov 20: Samuel/Sun, Murdoch, Ho/Wai, Paterson, Kou/Ryoo
- Nov 25: Valieva, Simpson, Engelking, Arajs/Hussain, Steiner/Wollaston
- Nov 27: Ahmadian/Demirtas, Jaroszewski, Yuen/Kwong, Coulombe

Problem 6–1. (Incorporating draws)

Independent Problem. *Do not discuss this problem with anyone except for course staff.* Many games, such as tic-tac-toe, satisfy all the properties in the impartial combinatorial game model *except* that draws (ties) can occur. Each player prefers winning to drawing, and drawing to losing. One way to model such games is as follows.

Definition. A *impartial combinatorial game with draws* is a 4-tuple (P, p_0, A, p^*) where (P, p_0, A) is an impartial combinatorial game and $p^* \in P$. We interpet the game as follows: if the game is ever brought to position p^*, then the game immediately ends in a draw. Otherwise, the last person able to make a move wins, as usual.

We want you to extend the $\{\mathcal{N}, \mathcal{P}\}$-position analysis by creating a third type of position, a $\mathcal{D}$-position. The idea is that once the game is in a $\mathcal{D}$-position, neither player can guarantee a win, but either player can guarantee a draw.

Problem 6–1(a). (2 points)

Give rules that define how to label all positions as $\mathcal{N}$, $\mathcal{P}$, or $\mathcal{D}$.

Problem 6–1(b). (2 points)

Prove that your rules work by giving an optimal strategy that looks only at the positions' labels.

Problem 6–2. (Split and run : 4 points)

Independent Problem. *Do not discuss this problem with anyone except for course staff.* *Split and run* is an impartial combinatorial game. Initially, there are two piles of counters of sizes m and n. On your turn, you must take away one of the piles and split the other pile into two nonempty piles. So every position can be written in the form (i, j) where i and j are the sizes of the two current piles; for example,

$$A(3, 5) = \{(1, 2), (2, 1), (1, 4), (2, 3), (3, 2), (4, 1)\}.$$

You can easily see that the game always ends at position $(1, 1)$. Classify (with proof), for all pairs (m, n) of positive integers, whether the position (m, n) is a $\mathcal{P}$-position or an $\mathcal{N}$-position.[1]

Problem 6–3. (Game products)

Collaborative Problem. *You can work on this problem with up to 3 of the other students in this class, but you must write up your solutions independently (without copying) and you must list all of your collaborators.* Analogous to the definition of the game sum $G + H$, we define the *game product* of impartial combinatorial games G and H, denoted $G \times H$, as follows:

- $P(G \times H) = \{(p_G, p_H) \mid p_G \in P(G), p_H \in P(H)\}$
- $p_0(G \times H) = (p_0(G), p_0(H))$
- $A(p_G, p_H) = \{(p'_G, p'_H) \mid p'_G \in A(p_G), p'_H \in A(p_H)\}$

In words, to play $G \times H$ is like playing in both G and H at the same time, where moving in $G \times H$ means moving in G *and* H simultaneously[2]; once either component game is over, so is the product game. *Remark: this defines a semiring on games, e.g.* $(G+H)\times K = (G\times K)+(H\times K)$.

In each part **(a)**–**(c)** of this problem, we expect a proof if (i) or (ii) is the correct answer, and two examples if (iii) is the correct answer.

Problem 6–3(a). (2 points)

If G is $\mathcal{P}$ and H is $\mathcal{P}$, is $G \times H$ (i) always $\mathcal{P}$, (ii) always $\mathcal{N}$, or (iii) could be either?

Problem 6–3(b). (2 points)

If G is $\mathcal{P}$ and H is $\mathcal{N}$, is $G \times H$ (i) always $\mathcal{P}$, (ii) always $\mathcal{N}$, or (iii) could be either?

Problem 6–3(c). (2 points)

If G is $\mathcal{N}$ and H is $\mathcal{N}$, is $G \times H$ (i) always $\mathcal{P}$, (ii) always $\mathcal{N}$, or (iii) could be either?

Problem 6–4. (Subtraction games)

Collaborative Problem. *You can work on this problem with up to 3 of the other students in this class, but you must write up your solutions independently (without copying) and you must list all of your collaborators.* Let k and $s_1, s_2, \ldots, s_k$ be positive integers, and n a nonnegative integer. The *subtraction game* $S_n(s_1, s_2, \ldots, s_k)$ is an impartial combinatorial game defined as follows: we start with n coins, each player must remove s_1 or s_2 or $\ldots$ or s_k coins from the pile on their turn, and the last player to move wins. (For example, the 10-coin game is $S_{10}(1, 2)$.)

Problem 6–4(a). (4 points)

For each $n \geq 0$, determine whether $S_n(2, 4, 7)$ is a $\mathcal{P}$-position or an $\mathcal{N}$-position.[3]

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[1] You can get half credit by just explicitly listing the correct answer for all $m, n \leq 6$.
[2] In comparison, game sums are "… *G or H*".
[3] You can get half credit by just explicitly listing the correct answer for all $n \leq 16$.

Problem 6–4(b). (3 points, but hard)

Suppose k and $s_1, \ldots, s_k$ are fixed. Show that the set

$$\{n \mid S_n(s_1, \ldots, s_k) \text{ is a } \mathcal{P}\text{-position}\}$$

is *eventually periodic*. (Definition: a set S is eventually periodic if there exist integers p (the period) and s so that for all $i \geq s$, we have $i \in S \Leftrightarrow i + p \in S$.)

Problem 6–5. (Squebblecross : 4 points)

Collaborative Problem. *You can work on this problem with up to 3 of the other students in this class, but you must write up your solutions independently (without copying) and you must list all of your collaborators.* Consider the following Treblecross-like game. The game is played on an infinite 2-dimensional grid of squares. We assign player 1 the symbol X and player 2 the symbol O. The game is played as follows: when it is a player's turn to move, they put their symbol in any empty square. The first person to make a 2×2 square of their own symbols wins, and if this never happens, the game is a draw. Find a strategy for player 2 which will prevent player 1 from winning (so either player 2 wins, or the game goes on forever). (Note: this game is not impartial nor does it have finite horizon, but one of the techniques we've seen still apply.)

Problem 6–6. (Shape-tac-toe)

Fun Problem. *This problem is not for credit; don't hand in a solution.* The following game generalizes Squebblecross even further. We fix a finite set S of shapes. Each player marks an X or an O on their turn in some empty square, on an infinite 2-dimensional grid. The first player to make one of the shapes in S out of their own symbols wins. (So Squebblecross is an example, where $S = \{2 \times 2 \text{ square}\}$.) Show that player 2 does not have a winning strategy. Generalize to higher dimensions and other infinite grids.