Surname (please print): ______________________

First Name: ______________________

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University of Waterloo
Final Examination
Fall 2007

Course Number	CO 456
Course Title	Introduction to Game Theory
Instructor	David Pritchard

Date and Time	7:30–10 PM, December 14, 2007
Duration	2.5 Hours
Number of Pages	5 (including cover sheet)
Exam Type	**Closed Book** with 1 page of prepared notes.
Materials Allowed	Letter-size 2-sided note sheet in your own handwriting.
Instructions	1. Fill in the details at the top right. 2. Be sure to read *all* questions. They do not necessarily appear in increasing order of difficulty. 3. There are 8 questions in this exam; the 8th question is a bonus question. Write your answers in the space provided, and use the back of the previous page for additional space. 4. You may not use calculators.

#1. Short Answer (12 points)

Please circle "true" or "false".

(a) (3 points) Every strategic game with a finite number of players and actions has a mixed Nash equilibrium. (*Note, a pure Nash equilibrium also counts as a mixed Nash equilibrium.*)

True False

(b) (3 points) In an extensive game without simultaneous or chance moves, with a finite number of players and histories, if there are two distinct subgame perfect equilibria s and s', then for each player i the utilities $u_i(\mathcal{O}(s))$ and $u_i(\mathcal{O}(s'))$ for their two outcomes are the same.

True False

(c) (3 points) In a Nim game where the initial position has four piles of size 3, 6, 9, 12, the first player has a winning strategy.

True False

(d) (3 points) In a potential game with a finite number of players and actions, best response dynamics always end in a global minimum of the potential function. (*Action profile a is a global minimum if* $\Phi(a) \leq \Phi(a')$ *for all* $a' \in A$.)

True False

#2. Mixed Equilibria Of Strategic Games (12 points)
Suppose that, in a 2-player strategic game, there exists a mixed strategy α_1 for player 1 and two mixed strategies α_2', α_2'' for player 2 so that both (α_1, α_2') and (α_1, α_2'') are Nash equilibria. Define a third mixed strategy α_2 for player 2 by

$$\text{for each } a_2 \in A_2, \quad \alpha_2(a_2) = \frac{\alpha_2'(a_2) + \alpha_2''(a_2)}{2}.$$

Informally, $\alpha_2 = (\alpha_2' + \alpha_2'')/2$. Show that (α_1, α_2) is a Nash equilibrium. (Hint: consider the Support Characterization.)

#3. Modeling Extensive Games (9 points)
The Team 1 "Wonders" and Team 2 "Tooters" have advanced to the final game of the UW Blurnsball League! At the start of every game of Blurnsball, two decisions need to be made:

- One team *sends* (S) and the other *receives* (R); someone needs to decide which team does what.
- One team plays *eastwards* (E) and the other plays *westwards* (W); someone needs to decide which team does what.

Overall, Team 1 is ahead in the standings. The rules of Blurnsball state: "First, the team that is ahead in the standings makes one of the two decisions. Then, the other team makes the other decision." For example, Team 1 could pick "Team 1 is E and Team 2 is W," after which Team 2 could pick "Team 1 is S and Team 2 is R."

Each team attaches a value to each of the 4 possibilities, as shown in the following table. Each team wants its value to be as large as possible, and doesn't care about the value of the other team.

Team 1 gets	Team 2 gets	Value to Team 1	Value to Team 2
SE	RW	0	2
SW	RE	2	1
RE	SW	3	0
RW	SE	1	3

(a) (4 points) Model this scenario as an extensive game; specifically, give a picture of the game tree that includes the payoffs for each terminal history and indicates the player to move at each nonterminal history.

(b) (3 points) Find all subgame perfect equilibria of this game.

(c) (2 points) For each SPE found in part (b): in the outcome generated by this SPE, what decision does team 1 make, and what decision does team 2 make?

#4. Extensive Games With Simultaneous Moves (9 points)
Two twins have invested some of their money in a bond. The bond matures for four years; in year i the value of the bond is V_i dollars, where $V_1 = 100, V_2 = 300, V_3 = 500, V_4 = 700$.

In each year, both of the twins need to independently and simultaneously decide whether or not to "cash in" the bond. In year i:

- If exactly one twin decides to cash in, she gets $\$V_i$, the other twin gets \$0, and the game ends.
- If both twins decide to cash in, each one gets $\$V_i/2$ and the game ends.
- If neither twin decides to cash in, the game continues for another year. (Assume that in the 4th year, both twins *have to* cash in.)

(a) (4 points) Model this scenario as an extensive game with simultaneous moves.
(b) (3 points) Find all subgame perfect equilibria of this game.
(c) (2 points) For each SPE found in part (b): what is the outcome generated by this SPE, and what are the payoffs?

#5. Impartial Combinatorial Games (12 points)
The *antigrundy game* is an impartial combinatorial 2-player game played with piles of counters. The number of counters in each pile must be a positive integer. As usual, the last person to move wins the game. For this game, on your turn, you must do the following: *pick any pile and divide it into two or more piles of equal size.*

Let $\widehat{n}$ denote a pile of n counters in the antigrundy game. As an example, here is a sample run of the game starting from a single pile of 12 counters.

- Player 1 splits the pile of 12 into 2 piles each of size 6, leaving $\widehat{6} + \widehat{6}$.
- Player 2 splits a pile of 6 into 6 piles each of size 1, leaving $\widehat{6}+\widehat{1}+\widehat{1}+\widehat{1}+\widehat{1}+\widehat{1}+\widehat{1}$.
- Player 1 splits the pile of 6 into 3 piles each of size 2, leaving $\widehat{2} + \widehat{2} + \widehat{2} + \widehat{1} + \widehat{1} + \widehat{1} + \widehat{1} + \widehat{1} + \widehat{1}$.
- In each of the next 3 turns, a pile of 2 is split into two piles of size 1. Thus player 2 makes the last move, and wins.

(a) (3 points) Determine the set of all positive integers n for which $\hat{n}$ is a $\mathcal{P}$-position.
(b) (9 points) Compute $g(\widehat{60})$. (Hint: you don't need to consider 60 values, e.g., you don't need to compute $g(\widehat{7})$.)

#6. Potential Games and Pure Equilibria (12 points)

(a) (8 points) A *symmetric 2-action game* is an n-player strategic game defined by constants $p_1, p_2, \ldots, p_n$ and $q_0, q_1, \ldots, q_{n-1}$ in the following way:

- $A_i = \{P, Q\}$ for each player i
- For any action profile a let $\pi(a)$ denote the number of players i for which $a_i = P$; then we define the utility function for every player i by

$$u_i(a) = \begin{cases} p_{\pi(a)}, & \text{if } a_i = P; \\ q_{\pi(a)}, & \text{if } a_i = Q. \end{cases}$$

(Example: for $n = 3$, a symmetric 2-action game is as follows, where $p_1, p_2, p_3, q_0, q_1, q_2$ are constants. The 3-player Prisoner's Dilemma had this form.)

$p1\backslash p2$	P	Q
P	p_3, p_3, p_3	p_2, q_2, p_2
Q	q_2, p_2, p_2	q_1, q_1, p_1

$p3$: P

$p1\backslash p2$	P	Q
P	p_2, p_2, q_2	p_1, q_1, q_1
Q	q_1, p_1, q_1	q_0, q_0, q_0

$p3$: Q

Show that every symmetric 2-action game has a pure Nash equilibrium, by showing that it is a potential game. (If you are unable to solve the general case, you can obtain half credit by proving the special case $n = 3$.)

(b) (4 points) Find a strategic 2-player game with the following properties:

- $A_1 = A_2 = S$ for some set S of actions such that $|S| = 3$
- $u_1(x, y) = u_2(y, x)$ for all $x, y \in S$
- The game has no pure Nash equilibrium.

#7. VCG+Clarke (9 points)

In this problem we ask you to prove the result mentioned in class about multiunit auctions for the VCG mechanism. Specifically, suppose that there are k identical copies of an item for sale, and that each of n players wants to purchase one. (Assume $n \geq k$.) We can model this as the set of alternatives

$$\mathcal{A} = \{S \mid S \subset \{1, \ldots, n\}, |S| = k\}$$

where the alternative $S \in \mathcal{A}$ represents that each player $i \in S$ won one of the items. We assume that each player i attaches value 0 to not winning the item and a fixed value $b_i \geq 0$ to winning the item, so

$$V_i = \{v_i \mid \exists b_i \geq 0 : v_i(S) = b_i \text{ for } i \in S; v_i(S) = 0 \text{ for } i \notin S\}.$$

Determine (with proof) the social choice function f and payment functions p_i that result from applying the VCG mechanism with Clarke pivot payments to the social choice setting $(\mathcal{A}, V)$. You can ignore cases where ties occur if you want. Once you obtain your final answer, please restate it briefly in words.

#8. Bonus: Impartial Combinatorial Games With Chance Moves (5 points)

Consider the following variant of the 10-coin game, where there are 2 players as usual. At the start of the game, there is a pile of n coins. On your turn, you can take away either 1 or 2 coins from the pile. However, after each player's turn, with probability $\frac{1}{2}$, one of the remaining coins disappears (if any are left). When it is a player's turn to move and there are no coins left, they lose and the other player wins.

For example, it is clear that the first player can win when $n = 1$ or $n = 2$. However when $n = 3$, if both players play optimally, the first player wins with probability $\frac{1}{2}$.

Assuming optimal play by both players, what is the probability that the first player wins when $n = 10$?