#1. Pure Equilibria (12 points)

Consider the following voting game. Two candidates in an election, A and B, have a *different* number of supporters. (In class, we considered a similar game where A and B have the same number of supporters.) Specifically, suppose that A has k supporters and B has m supporters, where $k > m > 0$.

Each supporter has two choices: either they can cast a vote for the candidate which they support, or they can stay home and cast no votes. The candidate with the most votes wins: so either A wins, B wins, or A and B tie. The payoff to a supporter who does not vote is 2 if their candidate wins, 1 if their candidate ties, and 0 if their candidate loses. The payoff to a supporter who votes is $2 - c$ if their candidate wins, $1 - c$ if their candidate ties, and $-c$ if their candidate loses, where c is a fixed real number with $0 < c < 1$. In the following three parts, be sure to fully justify your answer.

(a) (4 points) Find all pure Nash equilibria in which the candidates tie.

(b) (4 points) Find all pure Nash equilibria in which a candidate wins by one vote.

(c) (3 points) Find all pure Nash equilibria in which a candidate wins by two or more votes.

(d) (1 point) Using parts (a), (b), and (c), what is the set of all pure Nash equilibria?

#2. Dominance and Nash Equilibria (12 points)

(a) (4 points) Prove the following statement: in a strategic game, if each player i has a weakly dominant strategy $\hat{a}_i$, then the action profile $\hat{a} = (\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n)$ is a pure Nash equilibrium.

(b) (4 points) Give a 2×2 game where each player i has a weakly dominant strategy $\hat{a}_i$, but there is a pure Nash equilibrium a^* such that for each player i, $a_i^* \neq \hat{a}_i$.

(c) (4 points) Prove the following statement: in a strategic game, if each player i has a *strictly* dominant strategy $\hat{a}_i$, and a^* is a pure Nash equilibrium, then $a_i^* = \hat{a}_i$ for each player i. *Definition: an action a_i is strictly dominant for player i if a_i strictly dominates each other action a_i' of player i.*

#3. Pure Equilibria and Voting Games (12 points)

In this problem we consider a variant of Hotelling's electoral model. As before, we model the votes as uniformly occupying the interval $[0, 1]$, and each player's action set is $[0, 1]$. As before, each voter votes for the nearest candidate, splitting equally amongst all k nearest candidates in the event of a k-way tie. However, we modify the utility functions. Namely, assume that *each player's utility function is equal to the fraction of votes they receive.* (So for example, a loser would prefer to deviate to get more votes even if they are still not winning.)

On the last page of the exam, we include a definition of how votes are calculated with some examples, for your reference.

(a) (6 points) Suppose there are exactly two players (candidates). Find an action profile that is a pure Nash equilibrium, and prove that your answer is correct.

(b) (3 points) Suppose there are exactly three players (candidates). Show that there is no pure Nash equilibrium in which all three candidates pick the same position.

(c) (3 points) Suppose there are exactly three players (candidates). Using the result of (b), show that there is no pure Nash equilibrium.

#4. Symmetric and Mixed Equilibria (12 points)

A crime is observed by n people, where $n \geq 2$. Each person would like the police to be informed but would prefer to have somebody else be the one that calls the police. Specifically, suppose that each person attaches the value v to the police being informed and bears the cost c when making the phone call, where $v > c > 0$. We can model this as an n-player strategic game where each player has the action set $\{C, D\}$ where C represents *call* and D represents *don't call*. Then the utility function for each player i is

$$u_i(a) = \begin{cases} v - c, & \text{if } a_i = C \\ 0, & \text{if } a = (D, D, \ldots, D) \\ v, & \text{if } a_i = D \text{ and for some } j,\ a_j = C. \end{cases}$$

(a) (9 points) Find a symmetric mixed Nash equilibrium of this game. (In other words, find a mixed Nash equilibrium α in which $\alpha_i(C)$ is the same for all players i.)

(b) (3 points) Consider the equilibrium you found in part (a). Suppose that v and c are fixed. Compute the probability that *nobody* calls. As n increases, does this probability stay the same, go up, or go down?

#5. Lemke-Howson Algorithm (12 points)

In this problem we ask you to use the Lemke-Howson algorithm to find a mixed Nash equilibrium of a 2-player game. We have already set up the initial tableaux and performed some pivots. *We guarantee that the game is non-degenerate, so the min-ratio rule should always have a unique winner.* In the most recent iteration, the variable s_6 was the one to leave the basis.

$$\begin{array}{lllll}
x_1 = \frac{1}{2} & & -\frac{1}{2}s_5 & & \\
x_2 = \frac{1}{4} & -\frac{3}{4}x_3 & & -\frac{1}{4}s_6 \\
s_4 = \frac{1}{4} & -\frac{5}{4}x_3 & +\frac{1}{2}s_5 & +\frac{1}{4}s_6
\end{array}
\qquad
\begin{array}{llll}
r_1 = 1 & & & -2y_6 \\
r_3 = 1 & & -3y_4 & -y_6 \\
y_5 = \frac{1}{4} & -\frac{1}{4}r_2 & & -\frac{1}{4}y_6
\end{array}$$

(a) (6 points) Continue pivoting until the complementarity conditions are satisfied.

(b) (2 points) What are the final values of x and y? What is the Nash equilibrium α^* corresponding to these values?

(c) (4 points) The game we are solving here has payoffs as given in the below table. For each action a_i of each player i, what is $u_i(a_i, \alpha^*_{-i})$? Verify that the Support Characterization holds for your solution from part (b).

$p1\backslash p2$	4	5	6
1	0,1	0,2	2,0
2	0,1	4,0	1,4
3	3,2	0,0	1,3

#6. Bonus: Two-Player Games and Linear Programs (6 points)

Suppose we fix a finite 2-player strategic game. Without loss of generality, let player 1 have action set $A_1 = \{1, 2, \dots, m\}$ and let player 2 have action set $A_2 = \{m+1, m+2, \dots, m+n\}$. Below we give a linear program which is defined in terms of the game's utility functions. It has one variable x_i for each action i of player 1, and an additional variable δ.

$$\begin{aligned}
&\text{maximize } \delta \\
&\sum_{i \in A_1} x_i = 1 \\
&\delta - \sum_{i \in A_1} x_i u_1(i, j) \leq -u_1(1, j) && \forall j \in A_2 \\
&x_i \geq 0 && \forall i \in A_1
\end{aligned}$$

(a) (2 points) Find the dual of this linear program. You may assume that both programs are feasible and bounded.

(b) (4 points) Using part (a) and strong LP duality, prove that exactly one of the following two statements holds:

- there exists a mixed strategy α_1 of player 1 which strictly dominates the pure strategy "1"
- there is a mixed strategy α_2 of player 2 to which "1" is a best (pure) response.