Population Games

Note: the terminology "population game" and "pseudo-Nash equilibrium" are my own. I am providing these notes to complement my explanation of such games in lecture, since I do not know of a good similar reference.

A *population game* consists of 1) a finite set $\mathcal{C}$ of *choices* and 2) for each $c \in \mathcal{C}$ a utility function u_c which maps action profiles to real numbers, with action profile defined as follows.

An action profile in a population game is represented by a vector f of choices where for each $c \in \mathcal{C}$, we have $f_c \geq 0$, and $\sum_{c \in \mathcal{C}} f_c = 1$. The intent is that f_c represents the *fraction* of the population choosing c.

A *pseudo-Nash equilibrium* is an action profile f such that, whenever $c, c' \in \mathcal{C}$ and $f_c > 0$,

$$u_c(f) \geq u_{c'}(f).$$

Informally, it means that nobody in the population has incentive to switch actions.

Remark: the idea is that we want to look at the Nash equilibrium of some game as $n \to \infty$. Getting into the exact details is complicated and pseudo-Nash equilibria are a slightly weaker, but much simpler concept. E.g., one person switching might make an ϵ difference that we ignore in pNE's.

Example: Adopting a new technology. Here $\mathcal{C}$ ={buy, don't buy} and

$$u_{\text{don't buy}}(f) = 0 \qquad \text{for any } f,$$

$$u_{\text{buy}}(f) = -1 + 2 \cdot f_{\text{buy}}.$$

We proved in class that the following three outcomes are pNE's:

(a) $f_{\text{buy}} = 1, f_{\text{don't buy}} = 0$

(b) $f_{\text{don't buy}} = 1, f_{\text{buy}} = 0$

(c) $f_{\text{buy}} = 1/2, f_{\text{don't buy}} = 1/2$

and as an exercise you may prove that no other pNE's exist.

Simple Routing

Suppose that a large population of users wants to get from point A to point B, and that there are k different roads that each person can take. We assume that each player wants their trip to take as little time as possible. We also wish to allow each road to have its own unique characteristics; we hence assume for the ith road, that there is a *delay function* d_i that takes a real number in $[0, 1]$ as input and outputs a real number. The intent is that $d_i(x)$ represents the time to travel along road i when a a fraction x of the population is using road i.

We can model this as a population game as follows.

- Choice set: we define $\mathcal{C} = \{R_1, \ldots, R_k\}$ where R_i represents the ith road.
- Utility functions: we define $u_{R_i}(f) = -d_i(f_{R_i})$.

So, each player is more happy when their delay is smaller, and all they care about is the delay on their own road.

Example. For simplicity, let us examine a version of the simple routing game where all roads are identical and (as one might assume) as traffic increases, speed decreases. Let $d(x) : [0, 1] \to \mathbb{R}$ be an arbitrary increasing function of x and define $d_1(x) = d_2(x) = \cdots = d_k(x) = d(x)$ for all x.

First, we claim that the action profile f defined by

$$f_{R_1} = f_{R_2} = \cdots f_{R_k} = 1/k$$

is a pNE. Indeed, since $f_{R_i} = d(1/k)$ for all i, we know that all players obtain the same utility $-d(1/k)$ in action profile f, and that there is no alternative road c' for any user to switch to obtain a benefit.

Second, we claim that no other pNE's exist. Notice that for any other f there must be some roads R_i and R_j such that $f_{R_i} > f_{R_j}$ (i.e., if not all f_c values are equal, two are unequal). But then

$$f_{R_i} > 0 \text{ and since } d \text{ is increasing } u_{R_i}(f) = -d(f_{R_i}) < -d(f_{R_j}) = u_{R_j}(f)$$

which contradicts the definition of a pNE (take $c = R_i, c' = R_j$).