Micro-Review

In class (November 8) we proved the following:

Lemma (Copycat)**.** If G is a combinatorial game, then $G + G$ is a $\mathcal{P}$-position.

Proposition ($\mathcal{P}$-ignorance)**.** If G and H are combinatorial games and G is a $\mathcal{P}$-position, then whoever has a winning strategy in H also has a winning strategy in $G + H$.

We defined the Grundy function $g()$ recursively by the formula

$$g(G) = \text{mex}(\{g(G') \mid G' \in A(G)\})$$

where mex is the "minimal excludant" function. We also gave the statement of the Sprague-Grundy theorem:

Theorem (Sprague-Grundy)**.** If G is an impartial combinatorial game then $G + *(g(G))$ is a $\mathcal{P}$-position.

Finally, here is the argument that led me to say "if you know how to play Nim, because of the Sprague-Grundy theorem, then you know how to play arbitrary sums of impartial games" (provided you know their Grundy numbers). Let $G + H + K + \cdots$ be some sum of games. Then all of the following games have the same winner:

$$\begin{array}{rl} G + H + K + \cdots & \\ (G + *(g(G))) + G + H + K + \cdots & \text{(by S-G Thm. } G + *(g(G)) \text{ is } \mathcal{P}\text{, then use } \mathcal{P}\text{-ignorance)} \\ (G + G) + *(g(G)) + H + K + \cdots & \text{(by rearranging)} \\ *(g(G)) + H + K + \cdots & \text{(by copycat } G + G \text{ is } \mathcal{P}\text{, then use } \mathcal{P}\text{-ignorance)} \\ *(g(G)) + *(g(H)) + *(g(K)) + \cdots & \text{(by repeating the argument).} \end{array}$$

The last game is just a game of Nim with piles of size $(g(G), g(H), g(K), \dots)$ so if we know who wins that, we know who wins $G + H + K + \cdots$. We'll see later, using the same ideas, how to actually *find* a winning strategy in $G + H + K + \cdots$.

Proof of Sprague-Grundy

Now that we have some motivation, we can get to the proof of the Sprague-Grundy Theorem. It is useful to point out the following facts beforehand.

Fact. For any impartial combinatorial game G,

(a) Every option G' of G (i.e., every $G' \in A(G)$) has $g(G') \neq g(G)$.

(b) For each integer t with $0 \leq t < g(G)$, there exists an option G' of G with $g(G') = t$.

The proof of these facts is simple: look at the definition of $g()$.

A *null game* is a game G with $A(G) = \emptyset$. We leave it as an easy exercise to show that the sum of two null games is null.

Proof of Sprague-Grundy Theorem. Like the proofs of the Copycat Lemma and $\mathcal{P}$-ignorance, we use induction on depth(G).

Base case: depth(G)=0. Then G is a null game, as is $g(G) = 0$, and $G + *(g(G))$ is also null. Since null games are $\mathcal{P}$-positions, we are done.

Inductive case: This involves showing that every option $H \in (G+*(g(G)))$ is a $\mathcal{N}$-position, which in turn requires showing that every such H has an option $K \in A(H)$ so that K is a $\mathcal{P}$-position. Essentially, we are giving a winning strategy for player 2 in $G + *(g(G))$.

What options does $G+*(g(G))$ have? Either a move can be made in the left component G, or in the right component $*(g(G))$. Using Fact **(a)** we can further break down the left moves into two types depending on whether the Grundy value increases or decreases. This leads to case analysis. If $H \in A(G+*(g(G)))$, one of the following cases holds:

decreasing left move: H is of the form $G'+*(g(G))$ where $G' \in A(G)$ and $g(G')$ **is less than** $g(G)$. Notice that $K := G'+*(g(G'))$ is an option of H — we're just removing some Nim counters. Re-using the Sprague-Grundy theorem by induction, we see that K is a $\mathcal{P}$-position, as required.

reversible left move: H is of the form $G'+*(g(G))$ where $G' \in A(G)$ and $g(G')$ **is greater than** $g(G)$. By Fact **(b)**, G' has some option, call it G'', with $g(G'') = g(G)$. Thus $K := G''+*(g(G))$ is an option of H. Again by induction, K is a $\mathcal{P}$-position.

right move: H is of the form $G+*(t)$ where $t < g(G)$. By Fact **(b)**, G has some option, call it G', with $g(G') = t$. Then $K := G'+*(t)$ is an option of H and again by induction, K is a $\mathcal{P}$-position.

So no matter what, every option H of $G+*(g(G))$ is a $\mathcal{N}$-position, and hence $G+*(g(G))$ is a $\mathcal{P}$-position, as claimed. □

What Comes Next: The XOR Rule for $g()$

In class I will show Bouton's theorem, or in other words, I will show how to play Nim perfectly. The main result is that a Nim position $*(a)+*(b)+*(c)+\cdots$ is a $\mathcal{P}$-position if and only if $a \oplus b \oplus c \oplus \cdots = 0$.

The next proposition (whose proof uses Bouton's Theorem) is a kind of restatement of the Sprague-Grundy Theorem. I would suggest in fact that it is more important to remember the XOR rule — since it actually tells us how to compute $g()$ values — than it is to remember the Sprague-Grundy Theorem, which is somewhat abstract.

Proposition (The XOR Rule)**.** $g(G+H) = g(G) \oplus g(H)$

Proof. The following 4 games are all $\mathcal{P}$-positions:

(a) $(G+H)+*(g(G+H))$

(b) $G+*(g(G))$

(c) $H+*(g(H))$

(d) $*(g(G))+*(g(H))+*(g(G)\oplus g(H))$

Why? Items (a)–(c) follow from the Sprague-Grundy Theorem, and item (d) follows from Bouton's Theorem, since $x \oplus y \oplus (x \oplus y) = 0$ for any x, y (we use $x = g(G), y = g(H)$).

Using $\mathcal{P}$-ignorance, the sum of all 4 games is also a $\mathcal{P}$-position. Using the Copycat Lemma the sum simplifies as follows:

$$[(G+H)+*(g(G+H))]+[G+*(g(G))]+[H+*(g(H))]+[*(g(G))+*(g(H))+*(g(G)\oplus g(H))]$$
$$= *\,(g(G+H))+*(g(G)\oplus g(H)).$$

This game is a $\mathcal{P}$-position. But we know from our analysis of two-pile Nim that $*(a)+*(b)$ is a $\mathcal{P}$-position iff $a = b$. Thus $g(G+H) = g(G) \oplus g(H)$, as claimed. □