Mechanism Design.

- "Inverse problem" of game theory
  → used to find properties of games
  → now, given a desired property, find the game..

Typical property: truthfulness / incentive-compatibity /
implementation in dominant strategies
→ each user has hidden info but reporting it truthfully
is weakly dominant.

We'll study:
- cake-cutting
- VCG: a truthful mechanism for generalized auctions
- Myerson: optimizing auctioneer revenue among truthful auctions
↳ extension when prior distributions of bidders are unknown

Related to Arrow's theorem:
- in an election, do you have incentive to report a false vote?

Cake-Cutting

If the sister cuts the cake and then takes one part for herself, leaving one for her brother, typical dialogue is
"You took a bigger slice!"
"No, they look the same size to me."
A successful / diplomatic parent would suggest that if the older sister thinks they are the same size, she should be willing to swap pieces, giving the "you cut, I choose" protocol.

This puts the responsibility of mechanically cutting it in half as a game-theoretic motivation (SPE).

Motivation: wills, legal teritorial disputes

However, you-cut-I-choose has an additional property:
- even if players' concept of value/utility differ (e.g. frosting vs. sprinkles), each can be guaranteed* $\geq \frac{1}{2}$ of the cake
  ↳ truthful: don't need to tell whole secret valuation to opponent.

*: explain why.

The problem becomes very interesting for 3 or more players; e.g. the following is unfair (even if everyone has the same valuations)
→ player 1 cuts into 3 pieces; p2 chooses; p3 chooses; p1 chooses.
Q: Problem?
A: Player 3 could get 0. Note: redistributing between p2 & p3 does not fix it.

We'll fix this but before we do, here is a series of stronger & stronger properties a mechanism could satisfy:

Fairness: for each i, $u_i(\text{piece}_i) \geq 1/n$.
Envy-free: for each i and each j, $u_i(\text{piece}_i) \geq u_i(\text{piece}_j)$
Exact: for each i, and each j, $u_i(\text{piece}_j) = 1/n$.

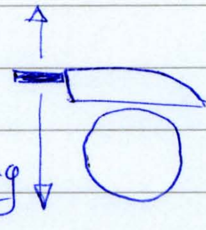
Assumptions: $u_i \geq 0$, $u_i(x \uplus y) = u_i(x) + u_i(y)$, u_i "continuous", ("monotone")

An exact division always exists (using $\leq n^2 - n$ cuts) by the Borsak-Ulam theorem. But finding one truthfully is harder.
(continuous protocol for $n=2$ (Austin), open for $n \geq 3$;
no discrete method with finite length, $n \geq 2$ (Robertson-Webb))

Envy-free: solutions known for any n (Conway; Brams-Taylor), but for $n \geq 5$ the finite number of cuts could be arbitrarily large.

Fair Cake-Cutting for n Players.

① Dubins-Spanier "moving-knife" protocol (1961)



— for each $i=1$ to n,

→ continuously move a knife over the cake remaining

→ each player should yell out "STOP" as soon as knife has passed over what they consider $1/n$ of the cake

↪ at time of first yell, cake is cut & that piece given to that player.

Claim: if ~~each~~ a player always yells out each time the knife has passed over what they consider as $\frac{1}{n}$, they get $\geq 1/n$ of the cake (in their valuation)

{"truthful"?}.

Proof: each round but the last, either they get $1/n$, or $\leq 1/n$ of cake is given to someone else. ☒

~~② Banach-Knaster discrete protocol (1944)~~

~~• in each of n rounds, each remaining player is asked to trim the current "piece" down to $1/n$ (or leave it alone, if it already looks like $\leq 1/n$ to them)~~

② Banach-Knaster discrete protocol (1944)

~~• n rounds~~

~~• in each round, consider all remaining cake as one "piece".~~

~~- each player gets to PA~~

• n rounds, in each one:

- ask one player to cut off a piece they consider $1/n$
- every other remaining player can "trim it" to what they consider $1/n$, if they think the piece is $> 1/n$ (else they pass).
- last player to trim it, takes it.

Again, truthful.

Ex. Find a "moving-knife" protocol for 2-player exact division.
(Hint: use two knives; or consider the cake as 2D.)

VCG Mechanism

Motivation: auctions for multiple items where bidders can submit arbitrary bids for every subset of items.

→ e.g.: will pay \$10 for both A and B, but one alone is worth \$1; will pay \$3 for X or Y; having both is still only worth \$3.

So as a strategic game,

$$A_1 = A_2 = \cdots = A_n = \{\text{all "bid functions" } 2^{\text{Items}} \to \mathbb{R}_+\}$$

We assume

$$u_i = \text{value}_i(\text{items awarded to } i) - \text{payment}_i$$

but the main challenge is that player i can lie: their reported action a_i is not forced to equal their value_i function.

Goal: design an allocation rule & payment rule so that bidding

$$a_i = \text{value}_i$$

is weakly dominant.

(we have already seen how to do this if $|\text{Items}| = 1$).

The main result of VCG (Vickrey-Clarke-Groves 61-71-73): such rules exist. It turns out to be easier to prove in a slightly more abstract form.

$\mathcal{A}$: set of all alternatives.

In an auction, $\mathcal{A} = \{\text{all allocations Items} \to \text{Players}\}$ (with each item assigned to at most one player)

<u>VCG Allocation Rule</u>: Pick ~~the~~ (an) alternative $a^* \in \mathcal{A}$ maximizing the <u>social welfare</u>

$$\sum_{i=1}^{n} \text{bid}_i(a^*)$$

~~VG~~

VCG Payment Rule: Player i pays the damage done to the other players,

$$\text{payment}_i = \max_{a' \in \mathcal{A}} \Big(\sum_{j \neq i} \text{bid}_j(a') \Big) - \sum_{j \neq i} \text{bid}_j(a^*)$$

{best other players could get} {what they actually got}

In proving the VCG theorem, one observation is useful: ~~they~~ player i has no control over the first term. We use this lemma:

Lemma. In a strategic game, if $\hat{a}_i$ is weakly dominant for player i, and we alter the payoff u_i for player i by ~~some function~~ ~~$h(a_{-i})$~~

$$u'_i(a) = u_i(a) + h(a_{-i})$$

where $h(a_{-i})$ is any function depending on players' choices $\neq i$, then $\hat{a}_i$ is still weakly dominant in the new game.

Proof. $\hat{a}_i$ weakly dominant in old game

$\Leftrightarrow \forall a_{-i}, \forall a_i, \quad u_i(\hat{a}_i, a_{-i}) \geq u_i(a_i, a_{-i})$

$\Leftrightarrow \forall a_{-i}, \forall a_i, \quad u_i(\hat{a}_i, a_{-i}) + h(a_{-i}) \geq u_i(a_i, a_{-i}) + h(a_{-i})$

$\Leftrightarrow \forall a_{-i}, \forall a_i, \quad u'_i(\hat{a}_i, a_{-i}) \geq u'_i(a_i, a_{-i})$

$\Leftrightarrow$ a_i weakly dominant in new game. $\boxtimes$

2nd term depends indirectly on bid_i due to a^*

Proof of VCG Theorem.

By lemma, ~~it suffices to show~~ we can ignore first $\max_{a' \in \mathcal{A}} \dots$ term. Then In VCG, a player with value_i and with bids_i, bids_{-i} leads to

① $a^* = \arg\max \sum_{i=1}^{n} \text{bid}_i(a)$

② $u_i = \text{value}_i(a^*) - \big(-\sum_{j \neq i} \text{bid}_j(a^*) + \{\text{ignored term}\}\big)$

$\cong \text{value}_i(a^*) + \sum_{j \neq i} \text{bid}_j(a^*)$.

Bidding truthfully ($\text{bid}_i = \text{value}_i$) makes a^* maximize the latter term. So, no false bid can cause u_i to increase. $\boxtimes$

Example: Auctioning k copies of an item.

$\mathcal{A} = \{\text{all } k\text{-subsets of } \{1, 2, \ldots, n\}\}$.

We restrict bidders to submit a single bid b_i which implicitly defines

$$\text{value}_i(a) = \begin{cases} b_i, & i \in a \\ 0, & i \notin a. \end{cases}$$

VCG Outcome $a^* = \arg\max_a \sum_i \text{value}_i(a) = \arg\max_a \sum_{i \in a} b_i$

- picks the k largest bids.

VCG Payment ... 0 for $i \notin a^*$, for $i \in a^*$ it is

$$\max_{a : i \notin a} \sum_{j \in a} b_j - \sum_{\substack{j \in a \\ j \neq i}} b_j$$

top k bids if we may not include i − top k bids, minus b_i. $= (k+1)^{st}$-largest bid.

Other General properties:

- payments ≥ 0 since $\max_{a' \in \mathcal{A}} \left(\sum \text{bid}_{-i}(a') \right) \geq \sum \text{bid}_{-i}(a^*)$

Exercise: Assume bids are nonnegative; show $\text{payment}_i \leq \text{bid}_i(a^*)$.

- by altering the "pivot function" $h(a_{-i})$ you can get other mechanisms with payments ≤ 0, e.g. competitive contract bids.

<u>Main Critique of VCG.</u>

Finding the arg max/max can be NP-complete! E.g. if bidders submit bids for all triples of items.

Also, it can be impractical to submit a bid of 2^{Items} numbers.

Leads to "combinatorial auctions" & interesting approximate results.

(See Nisan et al. §9.3 for more info)