Game Theory and Algorithms*
Lecture 11: Cake cutting; and the VCG mechanism

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Summary: In this lecture we introduce the "Inverse problem" of game theory. Given a set of properties we look for games satisfying them. We focus on two games, the cake cutting game, and the generalized auction. For such games, we analyze ways to force them have some properties we desire, like truthfulness.

1 Introduction: Mechanism Design

In this lecture we introduce the "Inverse problem" of game theory. So far given a game, we have been faced with the task of analyzing it. Now we do the reverse procedure, that is given a fixed set of properties we look for games satisfying them.

For an example, recall that the 2nd price auction was a regular auction where the winner (the highest bidder) would pay the 2nd highest price. This game is truthful, meaning that for each player, bidding their true value is a weakly dominant strategy. However, for the 2nd price auction, the players could get together to design a strategy in which their utility would increase. We say a game is group strategy proof, if the players cannot for a group strategy to improve their utilities. In this case, the 2nd price auction is not group strategy proof.

In this lecture we will study the following problems:

- Cake-cutting
- VCG: a truthful mechanism for generalized auctions
- Myerson: optimizing auctioneer revenue among truthful auctions.

2 Cake-cutting

Consider the familiar situation: two siblings, one brother and one sister, are dividing a cake. The brother cuts it into two parts, and takes one. The sister then protests, claiming that

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her share is smaller, while the brother says that she is lying. A Solomonic parent would suggest them to swap, giving way to the "yo- cut-I-choose" protocol. In general one could be dividing anything, be it heritage left by a parent, or territory, etc.

Note that the you-cut-I choose protocol has an additional property: even if plays concept of value differ (you could think that one of the siblings like the frosting more while the other prefers the sprinkles), each can be guaranteed at least half of the cake, in their own valuation. And this holds even in the case where they keep each other's valuation secret. Why? Consider the sibling cutting the cake, he can cut it into what he considers two equal parts, guaranteeing himself half of the value. While the other sibling can choose what for him is the best piece, getting at least half of his own valuation of the cake.

Notoriously, when the number of players increases to 3, things get quite more complicated.

Consider the following approach to dividing the cake: First player 1 cuts it into 3 pieces; then player 2 chooses a piece; afterwise player 3 chooses a piece; and finally player one gets the remaining piece. Is this procedure fair? Is it not, because player 1 could make two "empty" pieces, and hence player 3 would get zero. You could try to fix this, by saying that player 2 and 3 will play you-cut-I choose with the union of their pieces, but other problems will come up, can you think of one? Before proceeding this analysis, let us formalize some of our expressions.

For a cake-cutting procedure with n players, with utility functions u_i and where player i obtains piece$_i$ we say its *fair* if for each player i, $u_i(piece_i) \geq 1/n$. We say its *envy-free* if for each i and and j, $u_i(\text{piece}_i) \geq u_i(\text{piece}_j)$. We say that its *exact* if for each i, and each j, $u_i(\text{piece}_j) = 1/n$.

Note that while the you-cut-I-choose procedure is fair and envy-free, it is not exact.

Some assumptions:

- $u_i \geq 0$, $u_i(x \cup y) = u_i(x) + u_i(y)$, that is u_i is monotone.
- no atoms, i.e. a single point does not have a positive value.

By the Borsuk-Ulam theorem, an exact division always exists. But finding one is considerably harder

2.1 Fair Cake-Cutting for n players

2.1.1 Dubins-Spanier "moving knife" protocol (1961)

- Continuosly move a knife over the cake remaining.
- Each player should yell out "STOP" as soon as knife has passed over what they consider $1/n$ of the cake.
- The player that yelled out is assigned that piece (if multiple yells, choose anyone at random), and the procedure continues in the same fashion.

Claim 1. *If a player always yells out each time the knife has passed over what they consider as $1/n$, they will get at least $1/n$ of the cake (in their own valuaton).*

Proof. In each round, except the last, each player either gets $1/n$ or at most $1/n$ of the cake is given to someone else. □

2.1.2 Banach-Knaster discrete protocol (1944)

The following is done n times:

- A player cuts a piece which they consider to be of size $1/n$.
- Every other player can "trim it" to what they consider $1/n$ or else they can decide to pass.
- Last player to cut the piece, keeps it, the procedure continues with the remaining players.

This procedure is also truthful.

Exercise. Find a "moving-knife" protocol for 2-player exact division.

Hint: use two knives; or consider the cake as 2D.

3 VCG Mechanism

Motivation: auctions for multiple items where bidders can submit arbitrary bids for every subset of items.

Example 2. Consider an auction of two things that go well together, say bread and butter, someone might want to pay 10 dollars for both of them, but for only one they would only consider paying one dollar, since what they really want is to have breakfast. In another scenario, there might be two objects, say beer and wine, where the bidders are willing to pay 20 dollars for any one of them, but it is useless to get both, since they are having a dinner where they will either serve wine or beer, and hence they would be willing to pay only 20 dollars for both of them.

More formally, player i has the bid function $A_i : 2^{\text{items}} \to \mathbb{R}^+$. As usual, we assume $u_i = value_i(\text{items awarded to } i) - payment_i$. The main problem is that if we do not design a truthful game, it might be better for some players to lie, that is to claim a bid function that is not congruent to their own valuation.

Goal: design an allocation and payment rule so that bidding $a_i = value_i$ is weakly dominant. (Recall that when there is only one item, we had shown that the 2nd price auction was truthful).

Vicky-Clarke-Groves proved that a set of rules exist where bidding truthfully is weakly dominant. To prove this we introduce some notation.

Let $\mathcal{A} = \{\text{set of alternatives}\}$. Where by an alternative we mean a way of assigning the items to players (only one player per item, but a player might get more than one item).

VCG Allocation Rule: Pick an alternative $a^* \in \mathcal{A}$ maximizing the social welfare

$$\sum_{i=1}^{n} bid_i(a^*)$$

VCG Payment Rule: Player i pays the "damage" done to the other players,

$$\text{payment}_i = \max_{a' \in \mathcal{A}} \left(\sum_{j \neq i} bid_j(a') \right) - \sum_{j \neq i} bid_j(a^*)$$

The first terms stands for the best all players could get if player i was absent, while the second term is what they actually got.

Note that player i has no control over the first term, that is whatever they do, that term will remain unchanged. For the proof of the VCG theorem, the following Lemma will be useful.

Lemma 3. *In a stategic game, if $\hat{a}_i$ is a weakly dominant for player i, and we alter the payoff u_i for player i to*

$$u_i'(a) = u_i(a) + h(a_{-i})$$

where $h(a_{-i})$ is any function depending on the other players choices, then $\hat{a}_i$ is still weakly dominant in the new game.

The proof of this lemma is deferred to the next lecture.

Theorem 4. *VCG allocation and payment rules are truthful.*

Proof. Recall that for player i, its utility will be his valuation minus the payment, that is

$$u_i = \text{value}_i(a^*) - \max_{a' \in \mathcal{A}} \left(\sum_{j \neq i} bid_j(a') \right) + \sum_{j \neq i} bid_j(a^*),$$

where a^* is the alternative found by the VCG rule.

By Lemma 3 we can ignore the middle term. Hence, player i can only influence the outcome of a^*, and with that the third term. Therefore,

$$u_i \sim \text{value}_i(a^*) + \sum_{j \neq i} bid_j(a^*) = \text{ social welfare}$$

By the definition of a^*, this term is maximum when $\text{tvalue}_i = \text{bid}_i$. Hence no false bid can increase players i utility. □