Game Theory and Algorithms*
Lecture 8: Extensive games

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Summary: We introduce the concept of an Extensive Game, and some motivating examples. To easily show a difference, we discuss cheating in a strategic game as a game in extensive form. Further, we show how to write an extensive game in strategic form. We introduce the concept of subgame perfect equilibria (SPE), its relation with Nash equilibrium and an algorithm to find SPE.

Until now we dealt with strategic games, where players make their choices simultaneously. In the extensive game model, we will consider sequential structure, which will allow us to inspect situations where each player is free to change their decisions as events unfold.

1 Extensive games

- family of games allowing "sequentiality": one player moves (chooses an action), the another player moves (second mover can base their decision on first action)
- E.g. Cake-Cutting
  Player 1 cuts a cake into 2 pieces. Player 2 chooses a piece and takes it. Player 1 gets remaining piece. The utility of each player is the size of their piece.

Quick facts:

- In a finite extensive games without "ties" there is a natural unique solution concept: subgame perfect equlibria (SPE)
- when ties or infinite games are involved, SPE still makes sense but there may be none or several
- SPE can be found with the backwards induction algorithm

* Lecture Notes for a course given by David Pritchard at EPFL, Lausanne.

Example 1. (The Entry Game): Player 1 is an old, established firm, and player 2 is someone who considers entering the market (challenger). If player 2 decides to enter, the old company can either spend resources to fight or not. The tree representation of the game is shown in Figure 1.

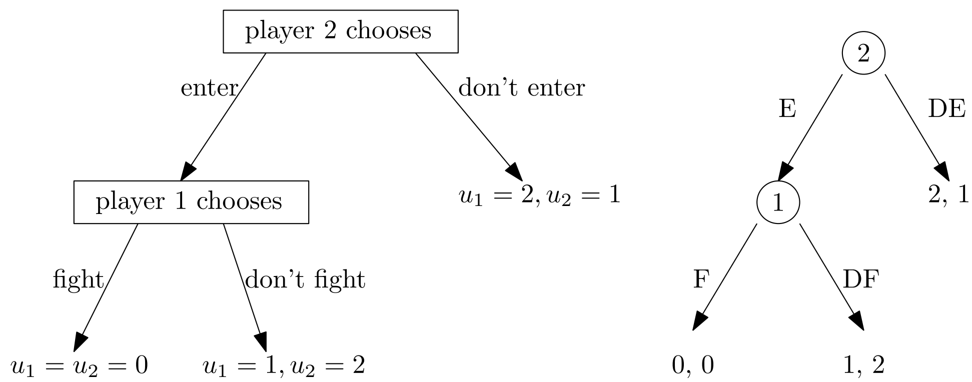

Figure 1: The Entry Game.

More generally an extensive game is defined by:

- n players
- a rooted tree
- each internal node is labelled by a player who moves
- each leaf node has utilities for each player

Gameplay: The root of tree is the initial state. At each internal node, the indicated player selects a child of current state. The state which represents a child becomes a new state. The end of game is when we reach a leaf node. Each player will get a utility assigned to the leaf.
Typical assumptions (as before): entire structure of game is known, players are greedy.

Let's analyze the Entry Game. Can we predict what will happen?

- if player 1 gets an oppurtunity to move, they will pick DF since they are greedy
- at start of game, player 2, knowing this, expects utility 2 if they pick E, and would get utlity 1 if they pick DE, so they'll pick E.

This is an example of using backwards analysis to find subgame perfect equlibria. It works the same way for any finite extensive game but runs into problems if there is ever a tie, (e.g. Figure 2) we can't make a good prediction.

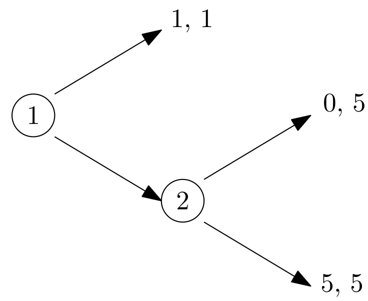

Figure 2: An extensive game with a "tie".

Exercise. Cannibalistic Lions.
Use backwards induction to predict the outcome of the following n-player game. Players are a hierarchy of n lions, who have found prey. Lion 1 must choose to either eat the prey or not. If she doesn't eat it, the game ends. Otherwise, lion 1 becomes fat and slow and lion 2 must choose to either eat lion 1 or not. If not, the game ends, otherwise lion 3 may eat lion 2 or not etc.
Preferences: being full $>$ being hungry $>$ being eaten.

Illustration: Cheating in 2x2 games. Imagine we are playing Bach or Stravinsky game (Figure 3), but player 1 gets to cheat by fixing their move to Bach before player 2 moves:

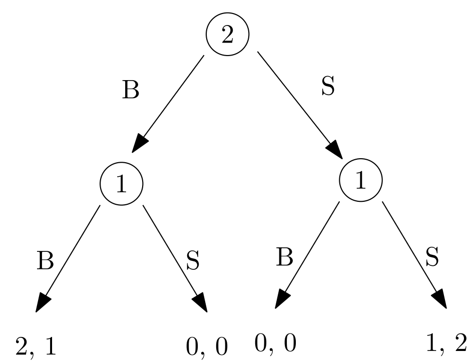

Figure 3: Bach or Stravinsky.

Q: What is the effect of this cheating?
A: It helps player 1: backwards induction on new game leads to the conclusion that they can expect a payoff of 2, previously they had no such guarantee.

Q: Is there a 2-player 2x2 game where this cheating (moving first) is bad?
A: Matching Pennies is one example: cheater always gets -1.

Q: What is the effect of cheating on the Prisoners' Dilemma?
A: No effect: everyone still picks D at every opportunity.

1.1 Writing Extensive Games in Strategic Form

Writing extensive games in strategic form will allow us to give an elegant definition of SPEs. Later, we will extend concept of SPE to ties, infinite choices and, simultaneous choices. Intuitively it is similar to NE: each player uses a best response in every subgame.

Definition 2. A strategy for player i consists of a valid choice of move for each node labelled i.
E.g. Figure 4.

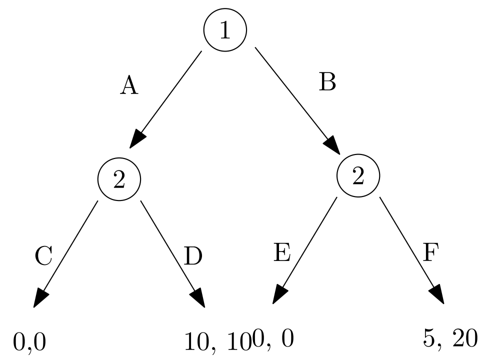

Figure 4: An extensive game

Player 1 has two strategies (A, B). Player 2 has four strategies (CE, CF, DE, DF).

We can convert any extensive game G to strategic form S in the following way:

- player set will remain unchanged
- A_i in S = all strategies of player i in G
- for strategy profile $s = (s_1, s_2, ..., s_n)$, the utility $u_i(s)$ in G is obtained from the leaf of G when all players play their choices in S.

p1/p2	CE	CF	DE	DF
A	$0^*, 0$	$0, 0$	$10^*, 10^*$	$10^*, 10^*$
B	$0^*, 0$	$5^*, 20^*$	$0, 0$	$5^*, 20^*$

Q: What are the NE of this game?
A: $(A, DE), (A, DF), (B, CF)$. In particular (B, CF) is a NE that gives utility of 20 to player 1. But backwards induction predicts the following:

- if player 1 picks A, then player 2 would pick D, $u = (10, 10)$
- if player 1 picks B, then player 2 would pick F, $u = (5, 20)$
- so player 1 will begin the game with A, $u = (10, 10)$

Why is (B, CF) a NE but not a plausible solution concept for this game? It relies on a non-credible threat: strategy CF punishes player 1 for picking A but it also punishes player 2 after the choice of A is made.

The previous example motivates the definition of subgame-perfect equilibrium. A subgame is a game obtained by looking at a subtree; this game has 3 subtrees (Figure 5):

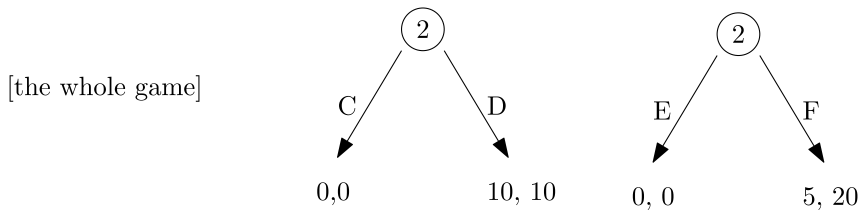

Figure 5: Subtrees.

Equivalently these are the games obtained when the players begin at a state h which is not necessarily the root.

For a state h let $h(s)$ be the leaf node reached starting at h and then assuming players play according to s.

Definition 3. A strategy profile s is a subgame perfect equilibrium if for all players i, all states h, and all other strategies s'_i for player i,

$$u_i(h(s)) \geq u_i(h(s'_i, s_{-i})).$$

I.e s must be NE in every subgame.

Comment: In the following game (Figure 6), AF is an interesting strategy for player 1.

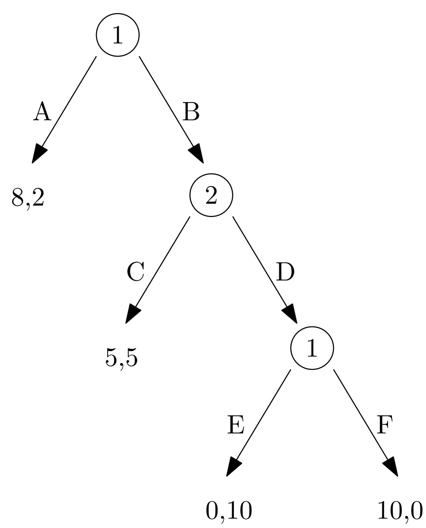

Figure 6: An extensive game.

If they play A, then why even decide between E and F in a subtree that is never reached? The definition of SPE gives the best justification: a strategy should specify what happens in all circumstances in order to justify threats. (e.g. saying that "(A, $\varnothing$) is an equilibrium" is not satisfying) Some people suggest a strategy should also have a "contingency plan" in case of errors.

Theorem 4. *In the absence of ties, backwards induction finds an SPE.*