Game Theory and Algorithms*
Exercises 1

To discuss in class on March 3, 2011

Note: there are a lot of problems here, you are not required to finish them all, but please challenge yourself! You should solve at least 3 of them well enough to be prepared to discuss them in class.

1 Lecture 1

Exercise. In IteratedElimination, it may happen at a given point in time that there are several possible choices of dominated strategies to eliminate. Prove that the final result of the algorithm is independent of the choice made in each iteration.

Exercise. (Challenge exercise) A *linear inequality system* $\{x \in \mathbb{R}^n : 0 \leq x_i \leq 1 \text{ for } i = 1, \ldots, n \text{ and } Ax \geq b\}$ consists of a matrix $A \in \mathbb{R}^{m \times n}$, a vector x of n real variables, and a vector $b \in \mathbb{R}^n$. (Each row A_i, b_i gives a linear constraint $\sum_i A_{ij} x_j \geq b_j$.) The simplex algorithm can be used to determine whether a given linear inequality system has any solution. Now, consider the *quantified* linear inequality system

$$\exists x_1 \in [0,1] \forall x_2 \in [0,1] \exists x_3 \in [0,1] \forall x_4 \in [0,1] \cdots : (Ax \geq b).$$

Give an algorithm to determine whether a quantified linear inequality expression of this form is true or false. (Hint 1: it won't be a polynomial-time algorithm. Hint 2: convert it to a game where the "$\exists$ player" tries to make the statement true and the "$\forall$ player" tries to make the statement false. Hint 3: show that the $\exists$ player can assume the $\forall$ player only ever chooses the values $\{0, 1\}$.)

2 Lecture 2

Exercise. Show that when we perform iterated elimination of strictly dominated strategies, all Nash equilibria survive (i.e., for every Nash equilibrium a and each player i, the algorithm never deletes action a_i).

* For a course given by David Pritchard at EPFL, Lausanne.

Exercise. In an election, there are $2k$ players: k of them support the candidate A, and k of them support the other candidate B. Each player can **V**ote or **A**bstain. The candidate with the largest number of voting supporters wins the election (they tie if they have the same number). Each voter gets a payoff of $+2$ if their candidate wins, 0 in a tie, and -2 if their candidate loses. Since the voters are lazy, their utility incurs a charge of -1 if they vote. (So each player's utility function can take on 6 values, the best is $+2$ (winning without voting), and the worst is -3 (voting and losing).) Find the Nash equilibria of this game.

Exercise. Consider the same game with 3 players, where A has 2 supporters and B has 1 supporter. What are the Nash equilibria? (Hint: it is qualitatively different from the previous exercise.)

Exercise. In Bertrand duopoly,

1. Show that (c, c) is a Nash equilibrium.
2. Show that (p_1, p_2) is not a Nash equilibrium if $p_1 < c$ or $p_2 < c$.
3. Show that (p_1, p_2) is not a Nash equilibrium if $p_1 = c$ and $p_2 > c$ (or vice-versa).
4. Show that (p_1, p_2) is not a Nash equilibrium if $p_1 > c$ and $p_2 > c$.

3 Lecture 3

Exercise. In Hotelling's game for two players, show there are no Nash equilibria other than (0.5, 0.5). (We already showed in class that (0.5, 0.5) is a Nash equilibrium.)

Exercise. In Hotelling's game for three players, show that there is no Nash equilibrium of the form (x, x, x); find a Nash equilibrium of the form (x, y, y).

Exercise. Show that the following *randomized* mechanism is *truthful in expectation* (i.e., if each player wants to maximize the expected value of their utility, then $a_i = t_i$ is a weakly dominant strategy): pick i uniformly at random from $\{1, \ldots, n\}$, and then set $p = a_i$. How does the expected *social welfare* $-\sum_i |t_i - p(t)|$ of this mechanism compare to the median mechanism?

Exercise. Consider a two-person auction with $v_1 > v_2$. Show that a first-price auction is not truthful, by showing that (v_1, v_2) is not a Nash equilibrium. Show that depending on how we break ties, there may be zero, one, or many NE.