Game Theory and Algorithms*

To discuss in class on March 22, 2011

You should solve at least 3 of these problems and be prepared to discuss them in class.

Lecture 4

Exercise. Use a polynomial-time linear programming subroutine to prove the following: there is a polynomial-time algorithm to determine whether a given game has any pure strategy that is strictly dominated by any mixed strategy.

Exercise. Find all mixed Nash equilibria of the Bach or Stravinksy game,

p1 \p2	B	S
B	2, 1	0,0
S	0,0	1, 2

Exercise. Find all mixed Nash equilibria of the following game:

$p1\backslash p2$	L	C	R
T	3, 4	5, 3	2, 3
M	2, 5	3, 9	4, 6
B	3, 1	2, 5	7, 4

Exercise. Consider the following game, the *Moose-Goose Hunt*. There are n players who are hunting a large moose. However, each one can either hunt the moose with the rest of the group, or choose to go off alone and hunt a goose instead. So $A_i = \{M, G\}$ for each player depending on what they choose to hunt. The utilities are given as follows, where $m > g > 0$ are fixed constants:

- any player who chooses G gets a utility of g
- if all players choose M, they all get a utility of m
- if player i chooses M, but at least one player chooses G, then player i gets 0 utility.

The idea is that hunting a moose is more profitable, but risky since it takes everyone to coordinate their efforts.

Find all *symmetric* mixed Nash equilibria of this game. (A mixed action profile is symmetric if each player assigns the same probability to M.)

* For a course given by David Pritchard at EPFL, Lausanne.

Exercise (Adapted from "Reporting a Crime," Osborne §4.8)**.** We have a game where n people witness a crime. Each one has the choice of either **R**eporting the crime or **N**ot reporting it to the police. Each player's payoff is affected by two factors: they prefer that the crime be reported by someone, but they have to pay a price to actually do the reporting. We model this as:

$$u_i(a) = \begin{cases} 0, & \text{if } a_1 = a_2 = \ldots, a_n = N; \\ v - c, & \text{if } a_i = R; \\ v, & \text{otherwise.} \end{cases}$$

Assume that $0 < c < v$, so the cost of reporting a crime does not exceed the benefit to each individual of it being reported.

- Find all pure Nash equilibria of this game; note none are symmetric.
- Find a symmetric mixed Nash equilibrium of this game (it is unique).
- Under this symmetric mixed Nash equilibrium, what is the probability that nobody reports the crime?
- As n increases, does this probability increase, decrease, or stay the same?

Lecture 5

Exercise. Consider *Matching n-ies*, the 2-player strategic game where both action sets A_i are $\{1, \ldots, n\}$; when $a_1 = a_2$ player 1 wins \$1 and player 2 loses \$1; when $a_1 \neq a_2$ player 1 loses \$1 and player 2 wins \$1. Find all mixed Nash equilibria of this game.

Exercise. Show that in a two-player zero-sum game, *every* mixed Nash equilibrium consists of a pair of maxminimizing strategies.

Exercise. Show that in the following 3-player zero-sum game, there is a mixed Nash equilibrium in which not all players are using maxminimizing strategies. (A maxminimizing strategy α_i is one which maximizes the value $\min_{a_{-i}} u_i(\alpha_i, a_{-i})$.)

$p1\backslash p2$	L	R
T	$-1,-1,2$	0,0,0
B	0,0,0	0,0,0

$p3$: X

$p1\backslash p2$	L	R
T	0,0,0	0,0,0
B	0,0,0	0,0,0

$p3$: Y

Exercise. Show that in the following 3-player zero-sum game, the three players' maxmin values do not add up to 0.

$p1\backslash p2$	L	R
T	$-1,0,+1$	$0,-1,+1$
B	$0,+1,-1$	$-1,+1,0$

$p3$: X

$p1\backslash p2$	L	R
T	$0,-1,+1$	$-1,0,+1$
B	$+1,-1,0$	$0,+1,-1$

$p3$: Y

Exercise (Due to Valentin Polishchuk)**.** In the figure below we give the schematic map of a museum with 5 rooms. A *guard* (player 1) and a *thief* (player 2) engage in the following game. Each simultaneously picks a room. If they pick the same room, or if the guard's choice of room is adjacent to the thief's, then the guard wins; otherwise, the thief wins. Model this as a 2-player zero-sum strategic game, using the utility value +1 to represent winning. Then, find a mixed Nash equilibrium. (Solve the LP in a computer algebra system or using a free online solver[1])

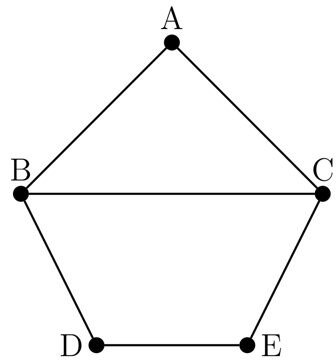

Figure 1: A map of the museum, with 5 rooms labeled A, B, C, D, E. We depict two rooms being adjacent by drawing a line segment to join the two rooms.

Lecture 6

Exercise. In lecture, we showed that the Lemke-Howson algorithm terminates at a $(x, y) \neq (\mathbf{0}, \mathbf{0})$. But, we actually need both that $x \neq \mathbf{0}$ *and* $y \neq \mathbf{0}$. Fix this hole in the proof.

Lecture 7

Exercise. Brouwer's fixed point theorem says that if $S \subset \mathbb{R}^d$ is compact (bounded & closed), convex and nonempty, and $f : S \to S$ is continuous, then there exists $x \in S$ such that $f(x) = x$. (We call x a *fixed point* of f.) Prove this theorem for $d = 1$.

Exercise. In this exercise we show that the conditions of Brouwer's theorem cannot be weakened. Give a counterexample (S, f) when:

- All conditions are satisfied, except that S is empty.
- All conditions are satisfied, except that S is not bounded.
- All conditions are satisfied, except that S is not closed.

[1] E.g., `http://www.neos-server.org/neos/solvers/lp:bpmpd/LP.html`

- All conditions are satisfied, except that S is not convex.
- All conditions are satisfied, except that f is not continuous.

Exercise (A class of games with pure equilibria)**.** Using Kakutani's theorem, prove the following theorem of Debreu-Fan-Glicksburg (1952). The setting is a game where each A_i is a closed convex nonempty subset of $\mathbb{R}^{d_i}$. We require that each u_i is a continuous *quasi-concave* function, meaning that the *level sets* $\{a \mid u_i(a) \geq C\}$ are convex for all i and all $C \in \mathbb{R}$. Prove that this game has a *pure* Nash equilibrium.

Exercise. Prove that the condition of quasi-concavity cannot be removed in the Debreu-Fan-Glicksburg theorem: consider the two-player zero-sum game with $A_1 = A_2 = [-1, 1]$ and $u_1 = a_1 a_2 + a_1^2 - a_2^2$; show it has no pure Nash equilibrium.

Exercise (Symmetric games have symmetric equilibria)**.** A game is *symmetric* if

- $A_i = A_j$ for all players i, j;
- $u_i(a_i, a_{-i}) = u_j(b_j, b_{-j})$ whenever $a_i = b_j$ and a_{-i} is a permutation of b_{-j}.

Show that every such game with $|A_i|$ finite has a mixed Nash equilibrium α with $\alpha_i = \alpha_j$ for all i, j.