Game Theory and Algorithms*
Exercises 4

To discuss in class on April 21, 2011

You should solve at least 3 of these problems and be prepared to discuss them in class. If you miss the class you should send one to three of the solutions to me in writing.

Lecture 11, 12

Exercise. Find a "moving-knife" protocol for 2-player *exact* division of a cake. (Hint: use two knives; or consider the cake as 2D.)

Exercise (Warm-up)**.** Assume that the bid functions in a VCG auction take on non-negative values for every alternative. Prove that for each player i, we have

$$\text{payment}_i \leq \text{bid}_i(\mathfrak{a}^*)$$

where $\mathfrak{a}^*$ is the alternative which was chosen (by VCG).

Exercise (Warm-up)**.** Consider the following example of the VCG mechanism. The government is proposing a new policy with three possible alternatives, X, Y and Z. There are three lobbying groups which have submitted bids on the three alternatives. If we run VCG on these bids, what alternative is chosen and what prices are charged?

	X	Y	Z
Lobby 1	12	9	2
Lobby 2	5	8	10
Lobby 3	4	5	6

Exercise. Suppose that we run the VCG mechanism on an auction of k identical items, except now we allow bidders to win more than one copy. Specifically, each bid b_i is a list of values $b_i(1) \geq b_i(2) \geq \cdots \geq b_i(k)$ where they would pay $b_i(1)$ for the first item, an additional $b_i(2)$ for a second item, etc., and we assume the bids $b_i(j)$ are nonincreasing in j. What alternative and does VCG select? In words, what prices are charged?

* For a course given by David Pritchard at EPFL, Lausanne.

Exercise. Consider an *interval* auction: we have a line of k objects, and for each $1 \le \ell \le r \le k$, each player submits a bid $b_i(\ell \dots r)$ indicating what would be the value if they won all items from ℓ to r, inclusive. (So each player submits $\binom{k+1}{2}$ numbers in total.) The auction is allowed to award several disjoint intervals $I_1, I_2, \dots, I_t$ to a player, in which case their presumed value is $\sum_{u=1}^{t} b_i(I_u)$. Show that the alternative with maximal social welfare can be computed in polynomial time. (Prices can be computed similarly.)

Lecture 13

Exercise. Consider auctioning a single item (i.e. there is always *exactly* one winner) to two bidders. The VCG auction/second-price auction awards the item to the highest bidder (breaking ties arbitrarily), and charges them the second-highest bid $b^{(2)}$. Show that for any truthful voluntary normalized mechanism, if its income is at least $b^{(2)}$ for all b, then the mechanism is a second-price auction.

Exercise. Now consider auctioning two items (we want $|\text{winners}(b)| = 2$ for all b), this time to three players. The VCG mechanism awards the items to the two highest bidders, and charges them each the third-highest price $b^{(3)}$. Find a truthful voluntary normalized mechanism such that (i) for all b, its income is at least $2b^{(3)}$ (ii) for some b, the winners don't correspond to the top two bids (so unlike VCG it does not maximize social welfare).

Exercise. What does Myerson's virtual VCG algorithm do if we have k identical items to auction away (and each of n players can win only one)?

Exercise. Suppose the auctioneer attaches a value $v_0 > 0$ to keeping the item: so instead of revenue they want to maximize their (expected) *surplus*, which equals income if they sell them item, and equals v_0 otherwise. Under the same assumptions as in Lecture 13, what truthful mechanism maximizes the expected surplus?

Exercise. Myerson's result proves that for the class of F's meeting all our assumptions, the revenue-maximizing auction for a single item among n players is a second-price auction with reserve x, where x does not depend on n. Show that this conclusion does not hold for *all* distributions. (Hint: one possible counterexample is a discrete distribution on a small number of values, where the optimal reserve price for 1 player is not equal to the optimal reserve price for 2 players.)

Exercise (Bulow-Klemperer)**.** We will compare two auctions. You can perform the first three parts independently of one another.

1. Under the assumptions from Lecture 13, show that the expected virtual valuation for player i is 0.

2. Show for any two independent random variables x, y, $\mathsf{E}[\max\{x, y\}] \ge \mathsf{E}[\max\{x, \mathsf{E}[y]\}]$.

3. Take F as described above; assume all bidders have independent valuations drawn according to F. Consider two second-price auctions: (i) we auction one item to n bidders, using the optimal reserve price; and (ii) we auction one item to $n+1$ bidders, using no reserve price. Show that the expected income of (ii) is at least as large as that of (i). (Moral: rather than be smart and compute the optimal reserve price, it's better to just attract one more bidder and use no reserve price.)

4. Generalize the previous result to k-item auctions: how many more bidders must you attract?

Lecture 14

Exercise. Show that for truthful-in-*expectation* mechanisms, there is no single-player mechanism which, for all b, gets an expected profit that is at least a constant factor of b.

Exercise. Show that the specific mechanism PROFIT-EXTRACT is group-strategyproof: if a subset I of players deviate from truthfulness so that one of their utilities increase, show that another of the players' utilities decrease.

Exercise. Prove that no randomized truthful mechanism for 2 players (for digital goods) has competitive ratio less than 2.