# Game Theory and Algorithms*
# Exercises 5 (Optional)

To discuss in class on May 23, 2011

## Lecture 15

**Exercise.** (a) Show that the Borda Count satifies the *Condorcet loser criterion*: if a candidate $A$ strictly loses in pairwise comparison with every other candidate, then $A$ is not the Borda winner. (b) Using part (a), show that Borda *never strictly opposes all pairwise comparisons*: there is always at least one pair of candidates $A, B$ such that at least half of voters prefer $A$ to $B$, and such that $A$ gets at least as many points as $B$ under Borda.

**Exercise.** Following up on the previous exercise: if $c$ determines a point-based voting system, and $c$ satisfies the *Condorcet loser criterion*, then prove $c$ is essentially equivalent to the Borda count, in the sense that $c_1, c_2, \ldots$ is an arithmetic sequence. (Hint: get a contradiction by focusing on a candidate who will violate the Condorcet loser criterion, and "average out" the other candidates.)

**Exercise.** Prove the stronger version of Arrow's theorem where we have 3 or more candidates, and where the output of the mechanism is allowed to be a weak order.

## Lecture 16

**Exercise** (Done in class)**.** Draw a directed graph whose vertex set is the candidates; whenever $A$ is preferred to $B$ by a majority of voters, draw a directed edge from $A$ to $B$. What conditions in this graph are necessary and sufficient for candidate $C$ to be electable by iterated pairwise elimination votes?

**Exercise.** In the Schulze method, show that if $p[A, B] > p[B, A]$ and $p[B, C] > p[C, B]$, then $p[A, C] > p[C, A]$.

* For a course given by David Pritchard at EPFL, Lausanne.

**Exercise.** Assume that the labels $d[\cdot,\cdot]$ are distinct; no label appears more than once. (One imagines this is likely to occur when the number of voters is sufficiently large compared to the number of candidates.) Show that the Schulze method produces no ties: $p[A,B] \neq p[B,A]$ for all $A, B$.

**Exercise.** (a) Give an example with three candidates where the output of the Schulze method gives $A > B = C = A$ (so the Schulze method gives a quasi-transitive order, but not a weak linear order). (b) Give an example with four candidates where the output of the Schulze method gives $A > B = C > D = A$.

**Exercise.** Consider the case that the candidates are nodes on an undirected **tree** (rather than a line). A voter $i$ is said to have *weakly single-peaked preferences* if for some node $r_i$, and all nodes $y$, and all nodes $x$ between $r_i$ and $y$ on the tree, voter $i$'s preferences amongst these pairs satisfy $r_i \geq_i x \geq_i y$. See Figure 1 for an example. Prove that there is a node $C$

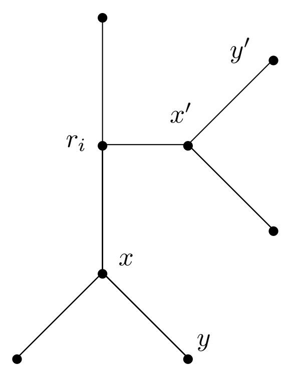


Figure 1: If a voter has weakly single-peaked preferences with peak $r_i$, and the tree of candidates is as shown, then voter $i$'s preferences must include $r_i \geq x \geq y$ and $r_i \geq x' \geq y'$.

such that for all $x$, at least half of voters prefer $C \geq_i x$ (a sort of weak Condorcet winner). Hints: the location of $C$ can be determined as a function of $r_1, r_2, \ldots, r_n$; it may be helpful to first solve the special case that the tree is a line.

**Exercise** (A false generalization)**.** If we allow weakly single-peaked preferences on a line, show Condorcet cycles can occur. Specifically, for three candidates on a line $A, B, C$, give weakly single-peaked preferences for some voters which give rise to a Condorcet cycle. (In this setting, directed edge $X \to Y$ means more voters prefer $X > Y$ pairwise than $Y > X$.)

**Exercise** (An algorithmic aside: Bartholdi & Trick)**.** If we are given the linear ordering of candidates, it is easy to check whether a given set of voter preferences are all single-peaked. However, what if we are not given the ordering of candidates? We could approach this by just testing all $k!$ possible orderings of candidates to see if any give rise to single-peaked preferences, but is there a polynomial-time algorithm? Show that the answer is "yes" by using a known polynomial-time subroutine for the following problem:

The **Consecutive-Ones** Problem

**Input:** A matrix of 0s and 1s

**Output:** Does there exist a permutation of the columns, so that in every row, all of the 1s are consecutive?

# Lecture 17

**Exercise.** Modify Pigou's example by replacing the bottom function with a power function, $\text{delay}_b(z) = z^c$. What lower bound does Pigou's bound give on the price of anarchy? How does this change as $c \to \infty$?

**Exercise** (Short but important)**.** Using the fact that the price of anarchy is at most 4/3, show that the factor 4/3 is the worst that can arise in Braess' paradox, for nondecreasing affine delay functions.

**Exercise.** Show that all potential functions $\Phi$ for a given potential game differ only by a constant.

# Lecture 18

**Exercise.** Show that if all edge delay functions are nondecreasing nonnegative affine functions, then the Price of Stability is at most 2. What bound can you get if each edge delay function is a quadratic of the form $z \mapsto a_e z^2 + b_e z + c_e$ for $a_e, b_e, c_e \geq 0$?

**Exercise.** Let $f$ be a socially optimal flow. The *unfairness* of the flow is the ratio

$$\max\{\sum_{e\in P} \text{delay}_e(f_e) \mid f_P > 0\} / \min\{\sum_{e\in P} \text{delay}_e(f_e) \mid f_P > 0\}.$$

What is the unfairness for a Nash flow? Show that for affine nonnegative nondecreasing delay functions, the unfairness is at most 2.

**Exercise.** In this exercise, you will prove the stronger bound of 4/3 on the price of anarchy for nonnegative nondecreasing affine delay functions.

1. Let $f^*$ be a flow. Pin the edge delays at constants $\text{delay}_e(f^*_e)$. Show that $f^*$ was an equilibrium flow for the original delays iff it is an optimal flow for the new constant delays. In other words, show $f^*$ is Nash iff for all other flows $f$,

$$\sum_e f^*_e \text{delay}_e(f^*_e) \leq \sum_e f_e \text{delay}_e(f^*_e).$$

2. Show that for any nonnegative nondecreasing affine delay function $d(z)$,

$$rd(r) \leq \frac{4}{3}(xd(x) + (r-x)d(r))$$

for $0 \leq r, x \leq 1$.

3. Show that for these delay functions, the price of anarchy is at most $4/3$.

**Exercise.** Consider any class of nonnegative nondecreasing delay functions which include all of the nonnegative constant cost functions. Let $\alpha$ be minimal such that for all functions $d$ from this class and all $0 \leq r, x \leq 1$,

$$rd(r) \leq \alpha(xd(x) + (r-x)d(r)).$$

By the previous exercise, the price of anarchy for instances with this class of delay functions is at most $\alpha$. Show that in fact the price of anarchy is *exactly* $\alpha$.

# Lecture 19

**Exercise.** *Split and run* is an impartial combinatorial game. Initially, there are two piles of counters of sizes $m$ and $n$. On your turn, you must take away one of the piles and split the other pile into two nonempty piles. So every position can be written in the form $(i, j)$ where $i$ and $j$ are the sizes of the two current piles; for example,

$$A(3,5) = \{(1,2),(2,1),(1,4),(2,3),(3,2),(4,1)\}.$$

You can easily see that the game always ends at position $(1,1)$. Classify (with proof), for all pairs $(m,n)$ of positive integers, whether the position $(m,n)$ is a $\mathcal{P}$-position or an $\mathcal{N}$-position.

**Exercise.** Let $k$ and $s_1, s_2, \ldots, s_k$ be positive integers, and $n$ a nonnegative integer. The *subtraction game* $S_n(s_1, s_2, \ldots, s_k)$ is an impartial combinatorial game defined as follows: we start with $n$ coins, each player must remove $s_1$ or $s_2$ or $\ldots$ or $s_k$ coins from the pile on their turn, and the last player to move wins. (For example, the 10-coin game is $S_{10}(1,2)$.)

1. For each $n \geq 0$, determine whether $S_n(2,4,7)$ is a $\mathcal{P}$-position or an $\mathcal{N}$-position.

2. Suppose $k$ and $s_1, \ldots, s_k$ are fixed. Show that the set

$$\{n \mid S_n(s_1, \ldots, s_k) \text{ is a } \mathcal{P}\text{-position}\}$$

is *eventually periodic*. (Definition: a set $S$ is eventually periodic if there exist integers $p$ (the period) and $s$ so that for all $i \geq s$, we have $i \in S \Leftrightarrow i + p \in S$.)

**Exercise** (Moore)**.** Find a Nim-like rule to determine winning and losing $(\mathcal{P}, \mathcal{N})$ positions in the following game: like Nim, there are several piles of counters, two players alternate turns and the last player to move wins; on your turn you can either remove any number of counters from one pile *and* (unlike Nim) you can, optionally, remove any number of counters from a second pile.

**Exercise** (Slither)**.** *Slither* is a game played on an undirected graph. The first player begins by placing a marker on any node $v_1$. Then, the second player slides the marker to any node $v_2$ adjacent to $v_1$. The players continue alternating, each time sliding to a vertex $v_{i+1}$ adjacent to $v_i$, and we also require that the $v_i$ are distinct, i.e. no node can be visited more than once. The first player unable to move loses.

1. Show that if the graph has a perfect matching, then player 2 has a winning strategy.

2. (Harder) Show that if player 2 has a winning strategy, the graph has a perfect matching.