Get To Know Your Trees

David Pritchard

Canadian Computing Competition, 2006 Stage 2

Outline

1. Preliminaries
2. Spanning Trees of Graphs
3. A General Framework
   - Depth-First Search
   - Breath-First Search
   - Minimum Spanning Tree
   - Dijkstra's Shortest Paths Algorithm
4. Advanced Tactics
   - A-Star, Meet in the Middle
   - Preorder, Postorder, Topological Sort
   - Biconnectivity, Strong Connectivity

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- . . . but we won't talk about this here.

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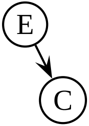

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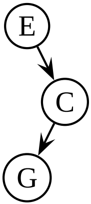

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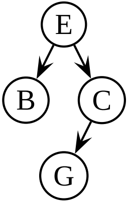

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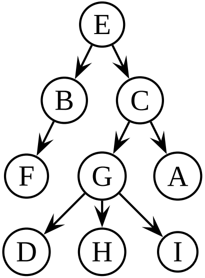

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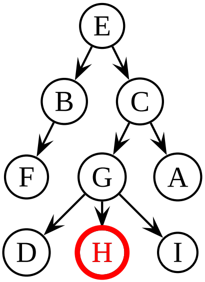

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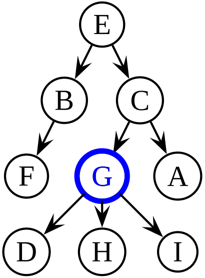

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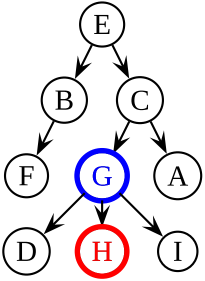

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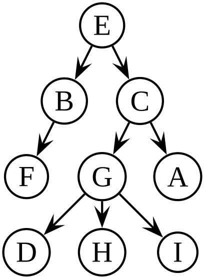

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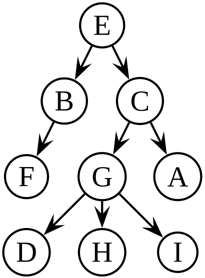

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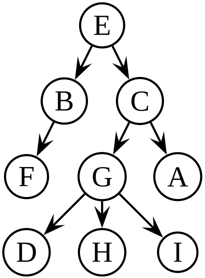

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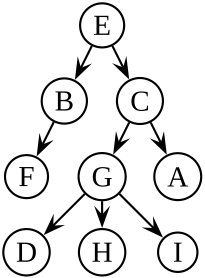

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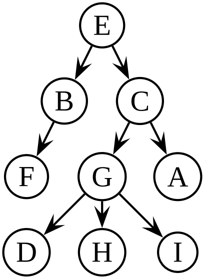

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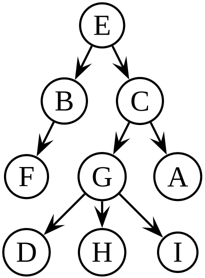
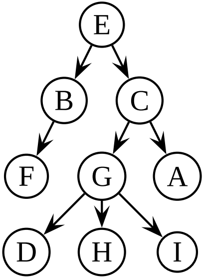
- occasionally, a tree (leaves = leaves, root = root)



Basic Botany

- Straightforward representation: keep an array P of the nodes' parents and an array C of child-lists.

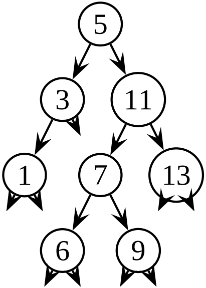
x	$P[x]$	$C[x]$
A	C	()
B	E	(F)
C	E	(G,A)
D	G	()
E	`nil`	(B,C)
F	B	()
G	C	(D,H,I)
H	G	()
I	G	()



- If we don't care about (or don't know) the order of each node's children then we may only need to keep track of P.
- Alternatively, we can just keep track of C.

Aside: Binary Trees

- Another form of tree is a *binary tree*.
- Each node may or may not have a left child, and may or may not have a right child.
- Each node is a record with fields (value, left, right), where left and right are pointers to nodes. A null pointer means that that child doesn't exist.
- If we stick values in the nodes the right way, we can make a *binary search tree* which is useful for some applications.
- Different generalization: k child positions is a *k-ary tree*.



Outline

Definitions

- A *graph* $G = (V, E)$ is a set V of n nodes (which we call $1, \ldots, n$) together with a collection E of *edges*. Each edge is just a pair of nodes.
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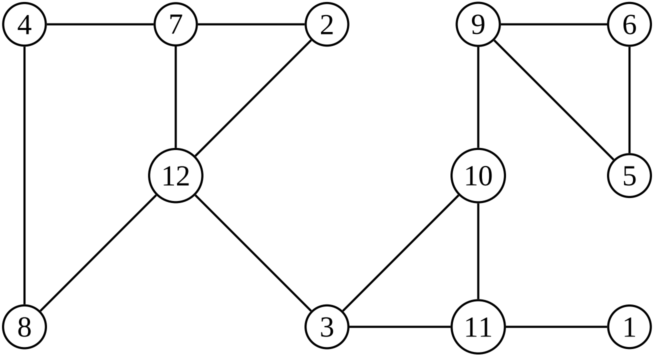

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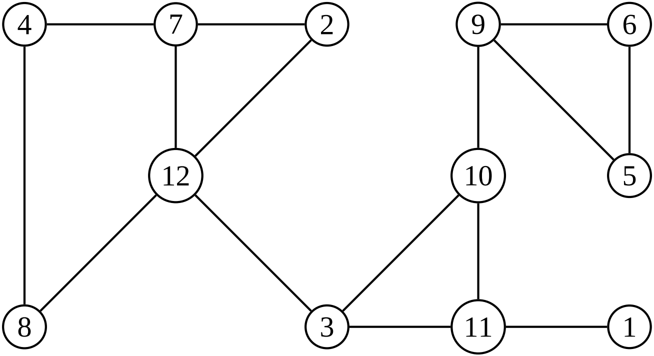

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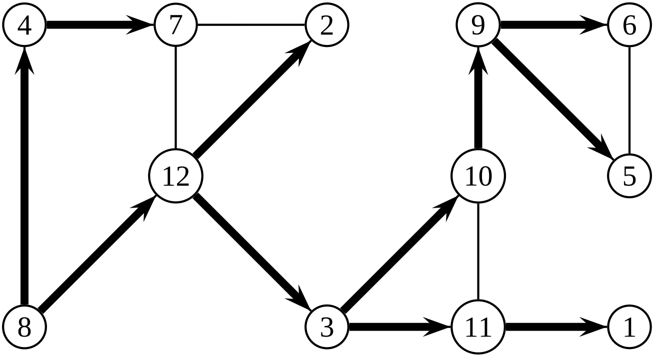

Spanning Tree ⇒ Check Bipartite

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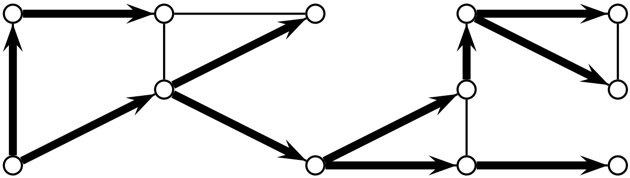

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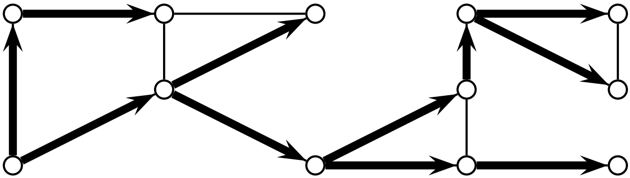

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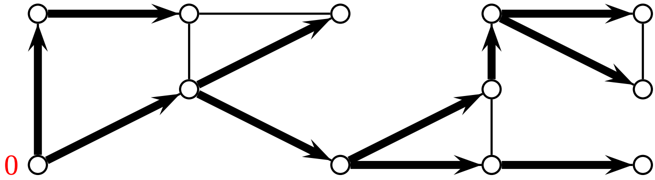

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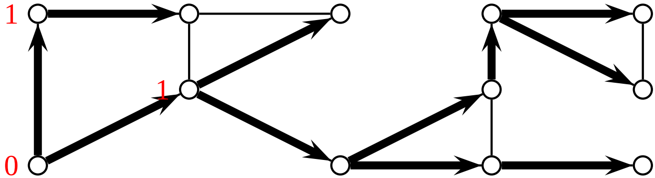

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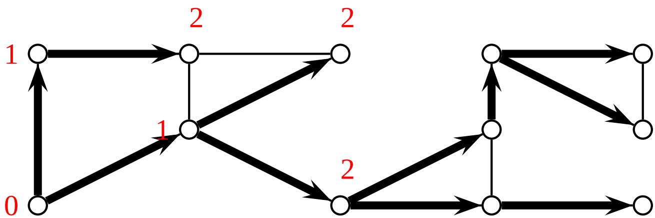

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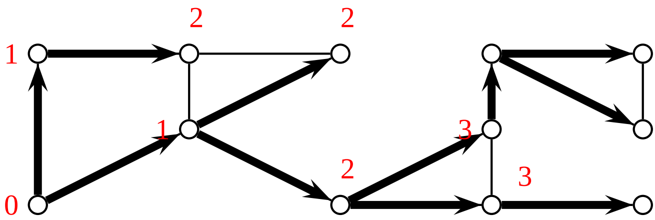

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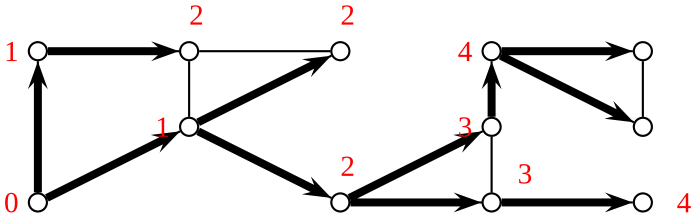

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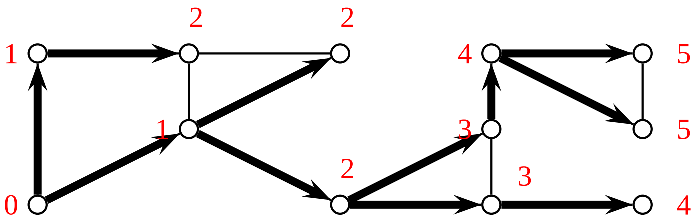

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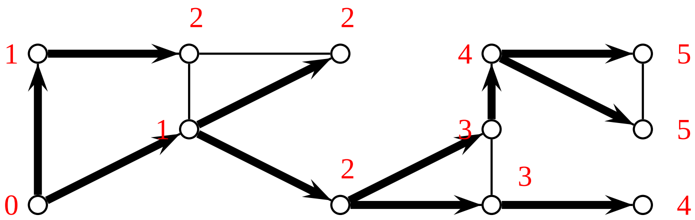

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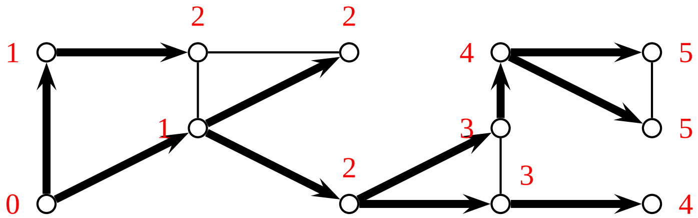

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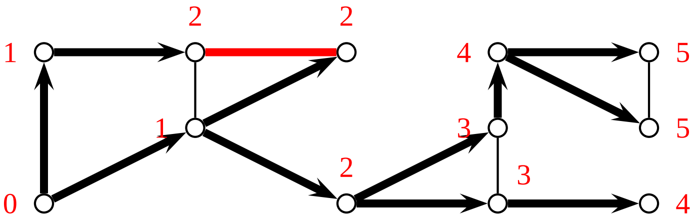

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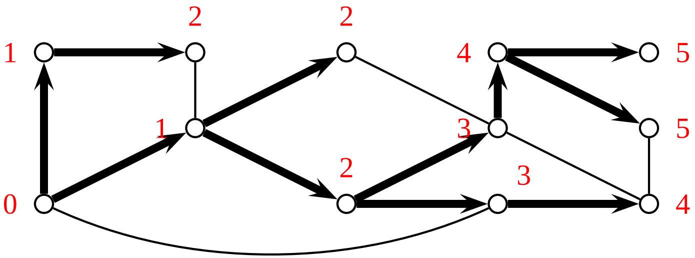

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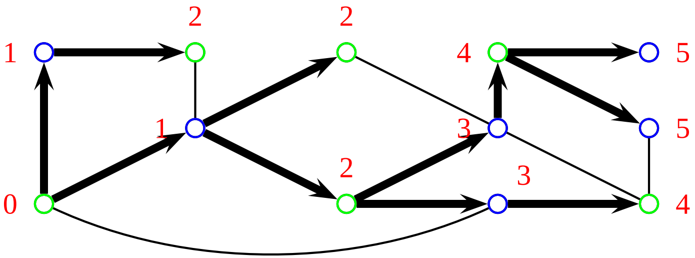
- We color the even-level nodes green and the odd-level nodes blue.



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- Implement priority queue with a *heap* or *balanced binary tree*.

Exploring a Graph

- Here is an abstract algorithm for exploring a graph.

```
procedure SEARCH-GRAPH(G, root)
    isExplored := boolean[vertices of G], initially false
    waitingEdges := struct⟨ pair⟨ vertex ⟩⟩
    waitingEdges.add((nil, root))
    while waitingEdges is not empty do
        (p, v) := waitingEdges.remove()
        if (!isExplored[v]) then
            isExplored[v] := true
            parent[v] := p
            for all neighbours w of v such that !isExplored[w] do
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- Basically we try to explore every edge that we learn about.
- No matter what order edges are removed from waitingEdges, we get a spanning tree.

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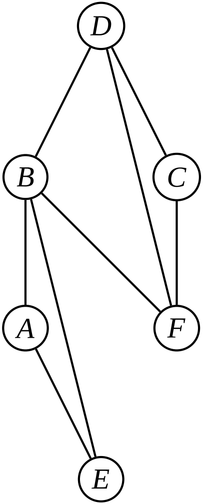

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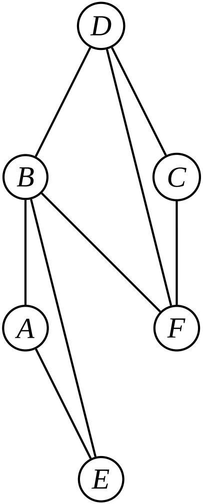

[Top] (`nil`, D)[Bot]

Depth-First Search

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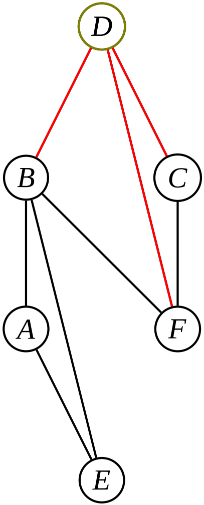

[Top] (D, B)(D, F)(D, C)[Bot]

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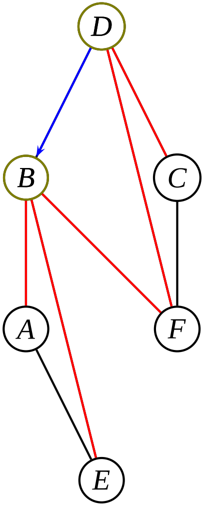

[Top] (B, A)(B, E)(B, F)(D, F)(D, C)[Bot]

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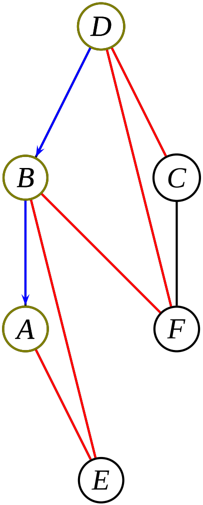

[Top] (A, E)(B, E)(B, F)(D, F)(D, C)[Bot]

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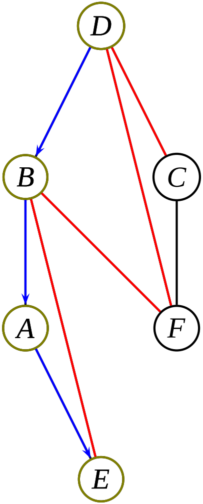

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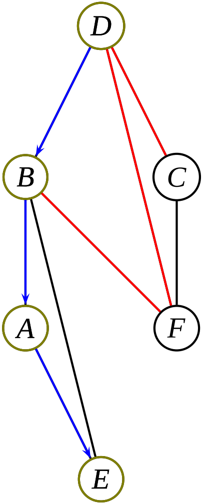

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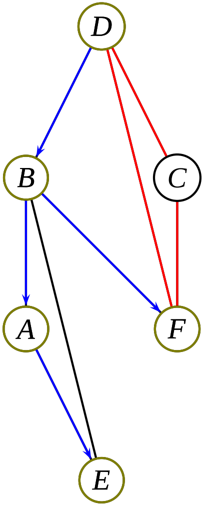

[Top] (F, C)(D, F)(D, C)[Bot]

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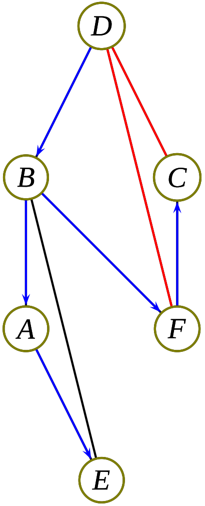

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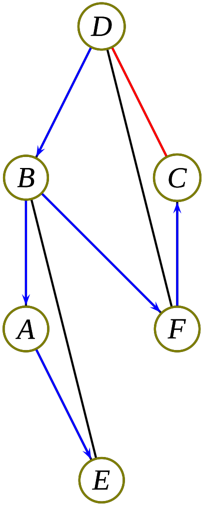
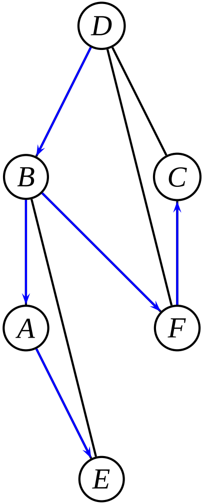


[Top] (D, C)[Bot]

Properties of Depth-First Search

- *Complexity*
- Each edge enters and leaves the stack exactly once.
- So $O(m+n)$ time complexity where $m = |E|, n = |V|$



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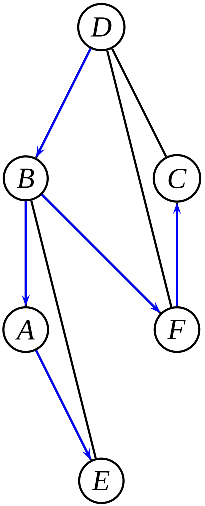

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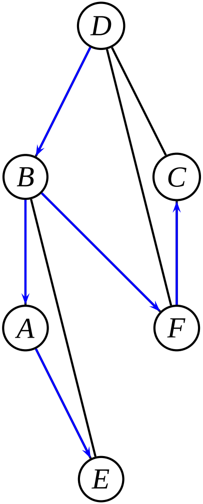
- *Applications*
- We will see later that using DFS and some other ideas (preorder, postorder) we can get efficient algorithms for biconnectivity and strong connectivity.



Outline

Breadth-First Search

- Make waitingEdges a queue: *breath-first search.*
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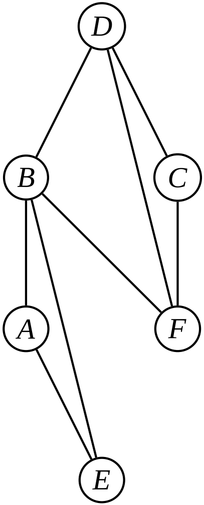

[Head] [Tail]

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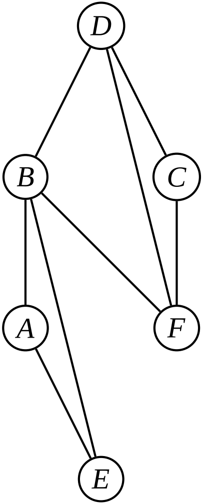

[Head] (nil, D) [Tail]

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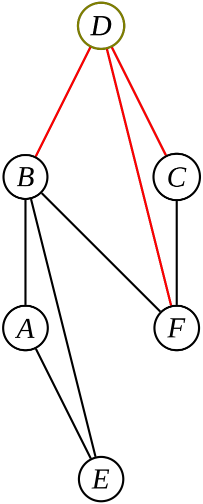


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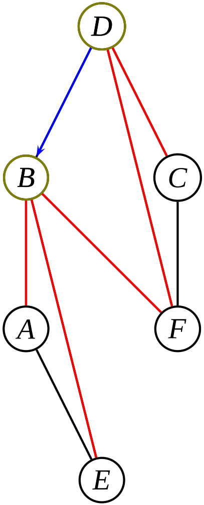


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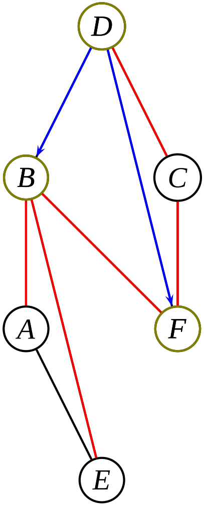

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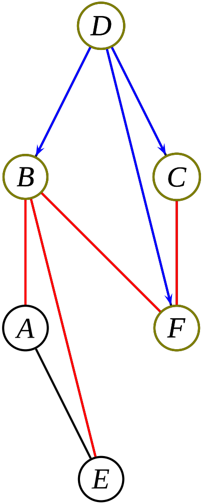

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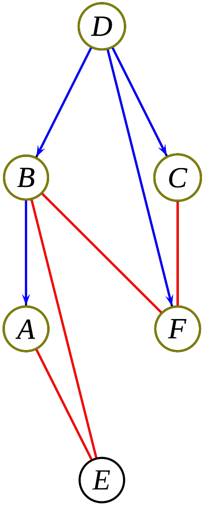

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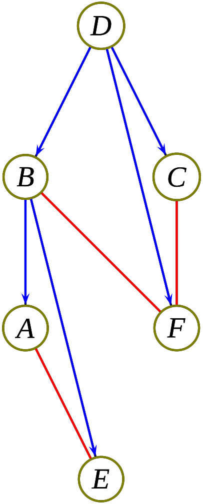

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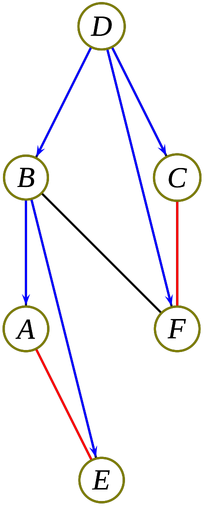

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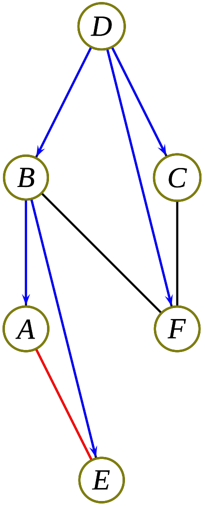

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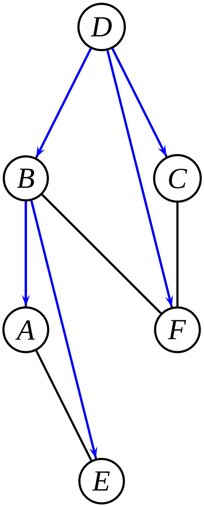

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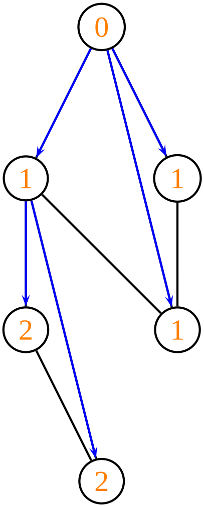

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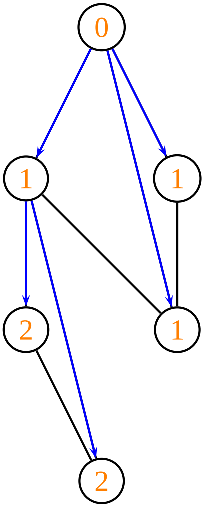
- *Girth* g : length of the shortest cycle.
- There is a length-g cycle through the root if and only if some non-tree edge uv satisfies level[u]+level[v]+1=g.
- So to compute g : do a BFS from each vertex and return the minimum value of level[u]+level[v]+1 over all non-tree edges uv in all trees.



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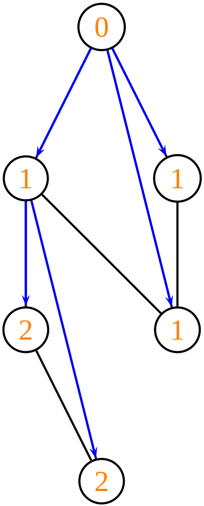
- So to compute g : do a BFS from each vertex and return the minimum value of level[u]+level[v]+1 over all non-tree edges uv in all trees.
- No known fast algorithm for determining the *longest* cycle.



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- Here's pseudocode for connected components. Search-and-Label(G, root) is any kind of search routine, but when it explores a node w it sets connected-component-label[w] := root.

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procedure CONNECTED-COMPONENTS(G)
    isExplored := boolean[v]        ▷ Assume vertices are 0, ..., v − 1
    for i := 1 to v − 1 do
        if !isExplored[i] then
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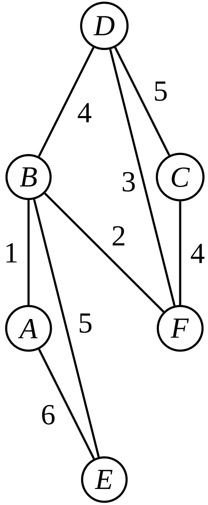
- Computes a *spanning forest* of G.

Outline

Minimum Spanning Tree

- Each edge uv is given a cost $c[u, v] = c[v, u]$.
- What spanning tree has minimal sum of edge costs?



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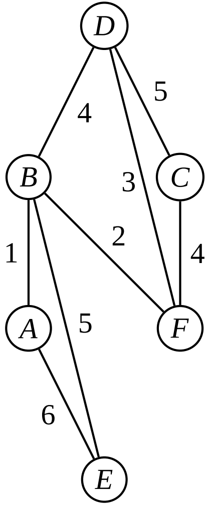

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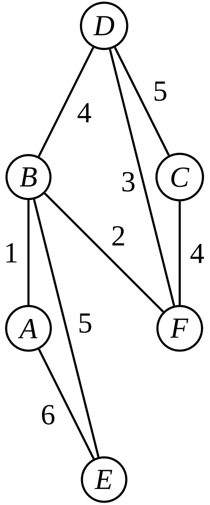

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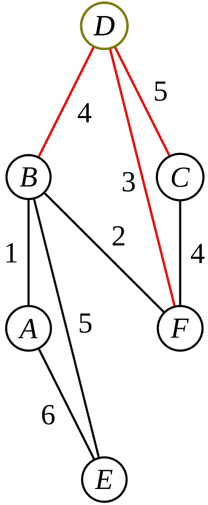

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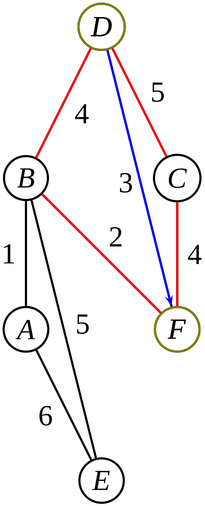

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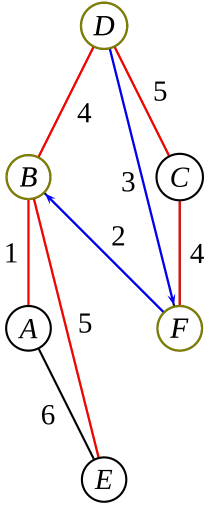

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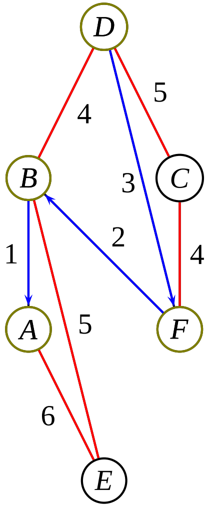

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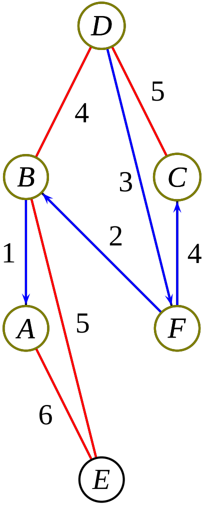

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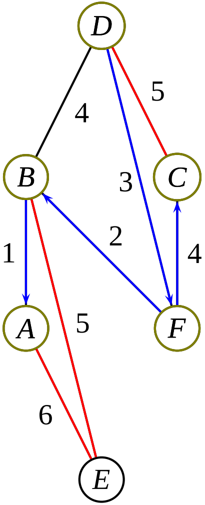

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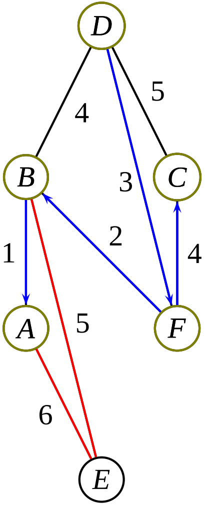

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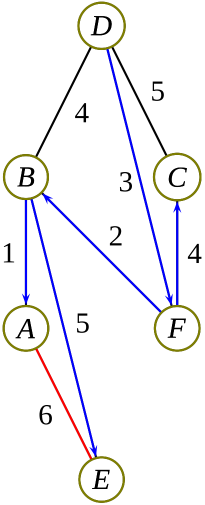

Minimum Spanning Tree

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- The priority queue always contains all edges that go *from* the current tree *to* some non-tree vertex.
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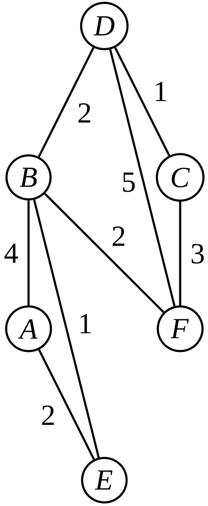
- Also known as Prim's algorithm.
- Can be implemented somewhat faster, in $O(m + n \log n)$ time.

Outline

Single-Source Weighted Shortest Paths

- Find the shortest distance d[x] from root to each vertex x in a *weighted* graph.



[Top]

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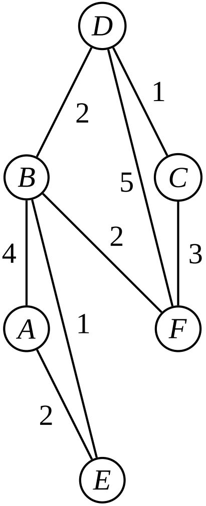

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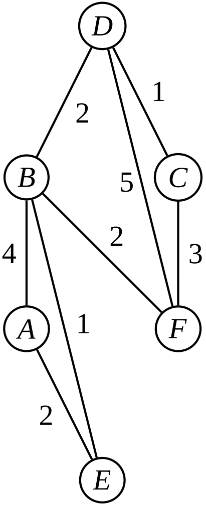

[Top] (nil, D)^0

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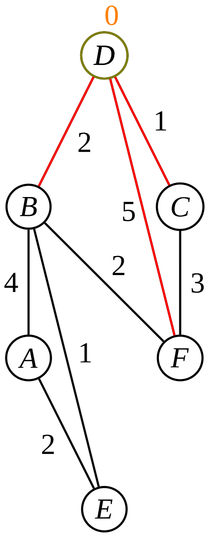

[Top] (D, C)^1(D, B)^2(D, F)^5

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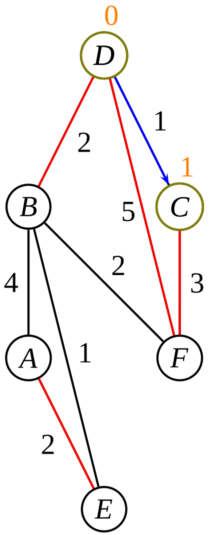

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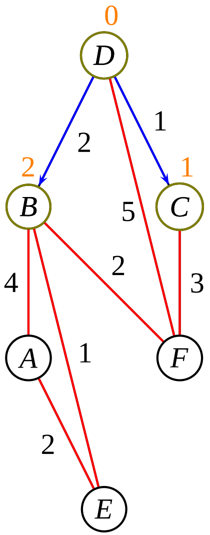

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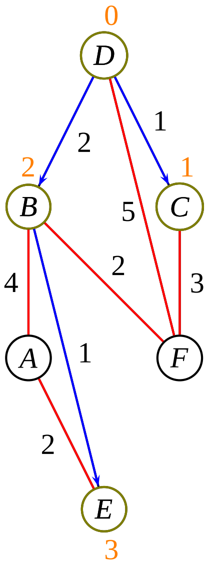

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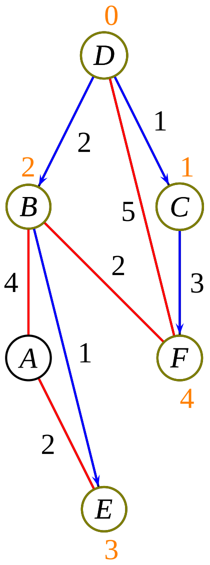

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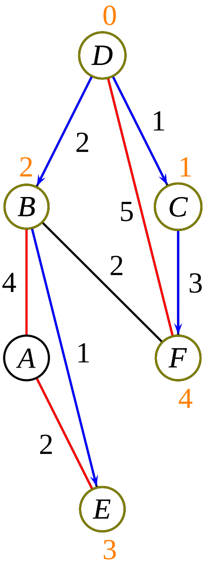

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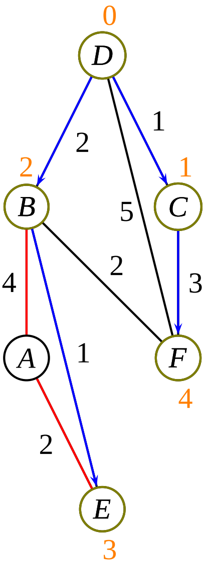

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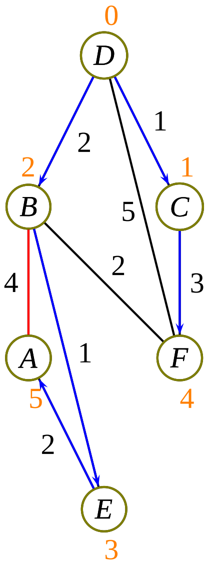

[Top] (B, A)[6]

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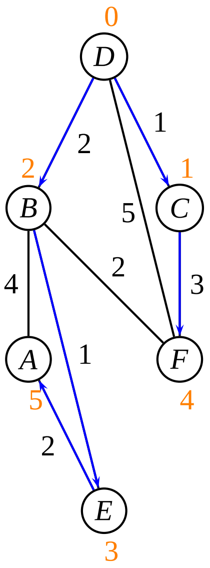

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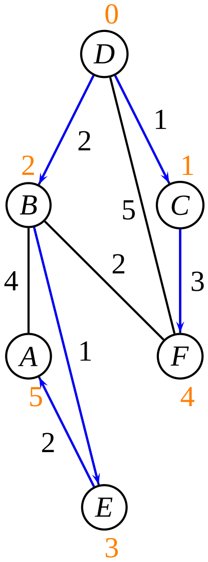

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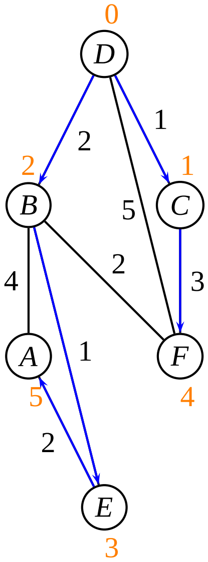
- Side note: $\mathrm{d}[u] \leq \mathrm{d}[v] + c[u, v]$ for all edges uv.



Single-Source Weighted Shortest Paths

- Is it somewhat surprising that the shortest paths can be even be *represented* by a tree?
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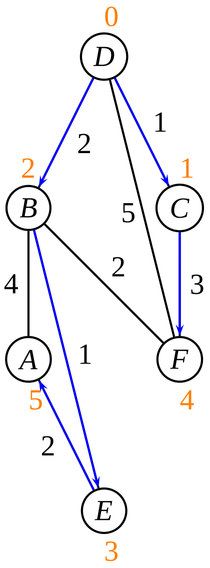

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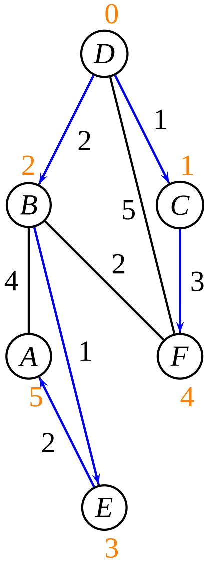
- Works for directed graphs. *Doesn't work* if there are negative edge weights.



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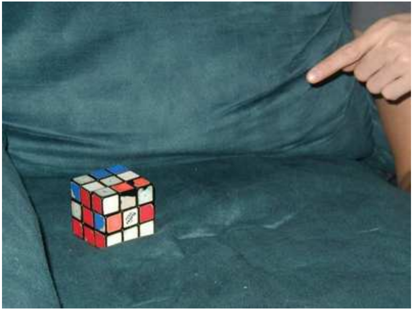

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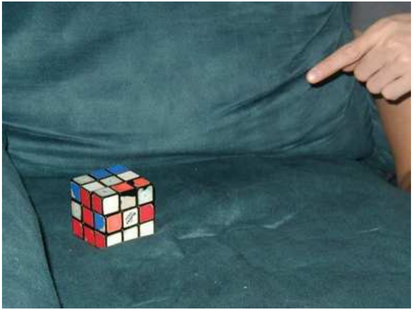

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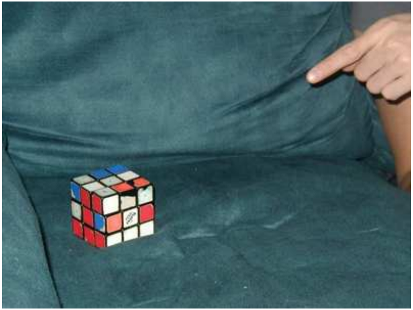

- Can we use BFS without actually constructing the whole graph?
- Sure. It is trickier to keep track of isExplored; you can use a Set class like a *hash set* or a *sorted set / balanced binary tree*.

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- 43,252,003,274,489,856,000 positions for a Rubik's cube.
- Actually, if x and y are diametrically opposite then the given strategy still would explore (nearly) the entire graph just to find the x-y path.
- But now I will explain 2 ways to improve performance even in this "worst" case: Meet-in-the-Middle and A* ("A-star") search.

Meet in the Middle

- Suppose that each node of our graph has k neighbours, and we are applying the previous BFS technique to find a shortest path between x and y who are at distance d.
- Roughly speaking, each level of the search will expand the universe of "explored" nodes by a factor of d, so about d^k total time is needed.
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- What's a simple way to improve? (Hint: look at the title)
- Conduct a BFS simultaneously from x and y.
- Think of the BFS from y as going backwards.
- Do a level of x's BFS, then y's BFS, then x's, etc.
- Let P be a shortest path between x and y, and z be a middle point of that path; hence it is distance $k/2$ from both x and y.
- We can detect that two trees will hit after $k/2$ rounds — total time complexity $O(d^{k/2})$.

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- If y is the unscrambled state, then we may take

$$h(v) = \text{number of out-of-position elements in } v.$$

- In route planning, h can be the Euclidean distance to y.

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- This *always* performs at least as quickly as Dijkstra's algorithm.
- As $h(v)$ increases towards a better underapproximation of $dist[v,y]$, the number of iterations required by A^* search decreases.

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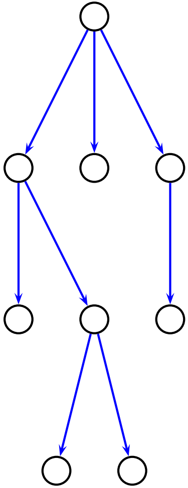
- Imagine a squirrel walking along the edges of the tree, starting with the root.



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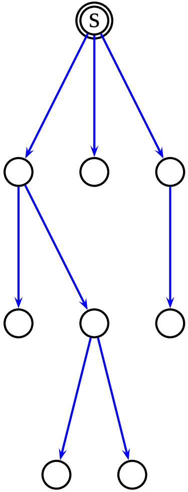

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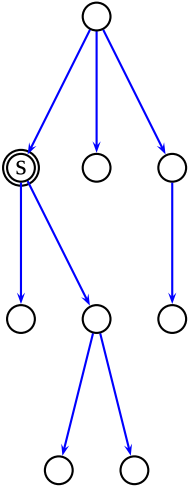

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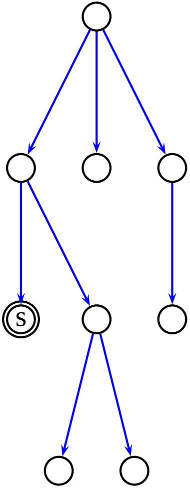

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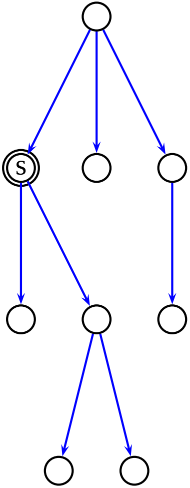

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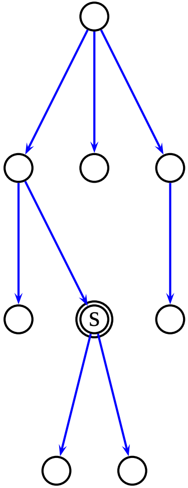

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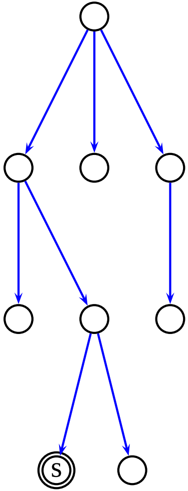

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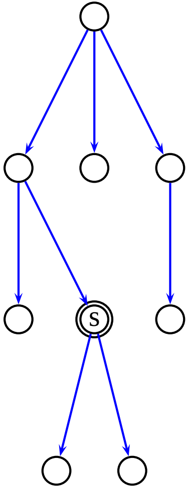

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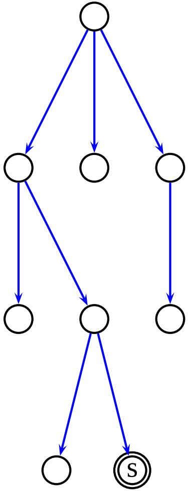

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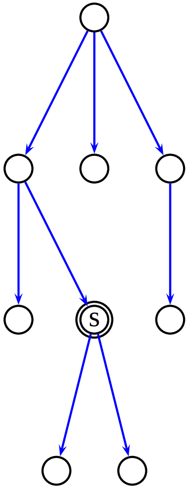

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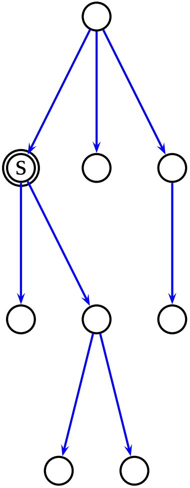

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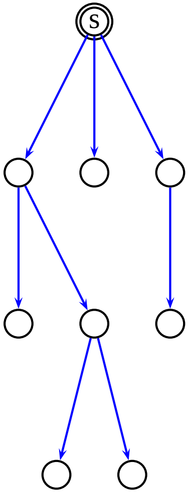

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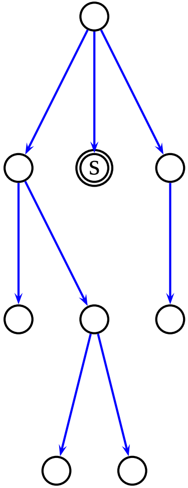

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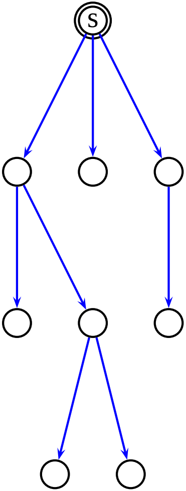

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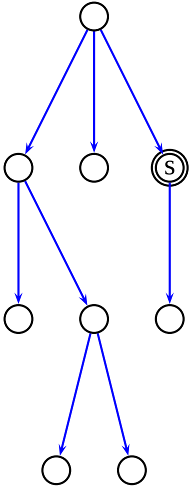

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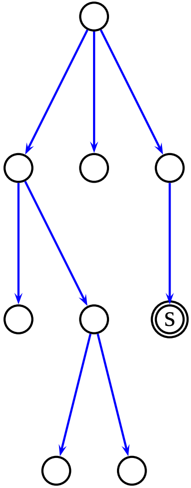

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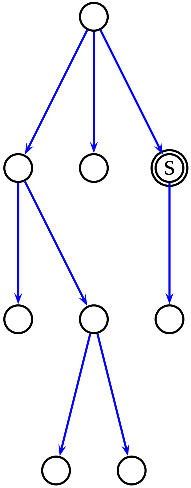

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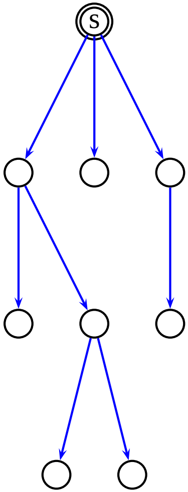

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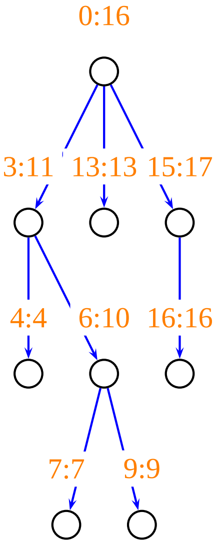
- For each node record the first and last time the squirrel visited the node.



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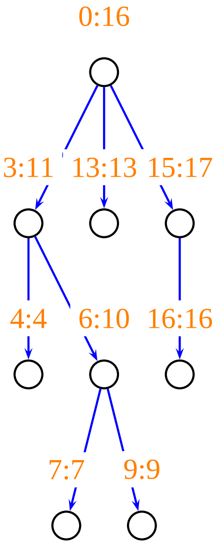
- Each vertex has preorder label less than its children.
- Conceptually: visit children of x *after* visiting x.



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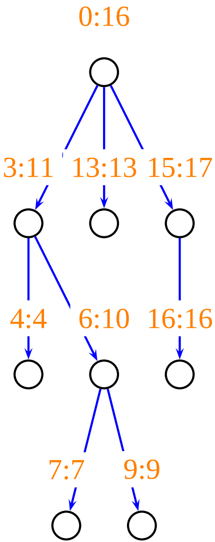

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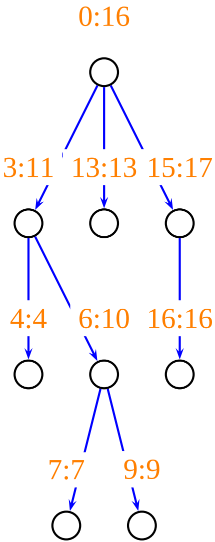

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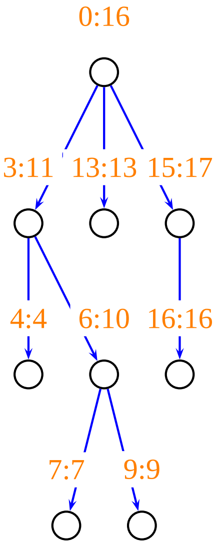
- *Aside:* for binary trees there is also *inorder* where you first visit the left child, then the root, then the right child.



Topological Sort

- *Topological sort* models the following problem.

[1] Usually DFS as it leads to efficient postorder computation.

Topological Sort

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- Same idea gives *cycle detection.*

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DFS Lite & Topological Sort

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DFS Lite & Topological Sort

- DFS is often implemented without an *explicit* stack.
- Here's a short implementation of topological sort:

```
isExplored := boolean[v]                          ▷ Initialized to false.
postList := list⟨ int ⟩                            ▷ Initially empty.
procedure DFS-ORDER(G, v)
    isExplored[v] := true
    //preList.add(v)
    for all outneighbours w of v do
        if (!isExplored[w]) then DFS-Order(G, w)
    postList.add(v)
procedure TOPOLOGICALSORT(G, v)
    for i := 0 to v − 1 do
        DFS-Order(G, i)
    return postList
```

Outline

Bridge-Finding

- An edge of a connected graph is a *bridge* if, when it is deleted, the graph is no longer connected.
- Equivalently uv is a bridge if every path from u to v uses the edge uv.
- How can we determine the bridges of a graph?

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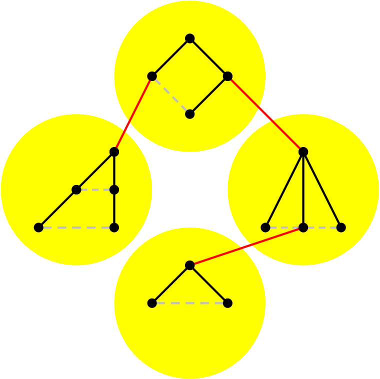
- It is clear that any spanning tree contains all bridges.
- Furthermore we can argue that the tree edge $(P[v], v)$ is a bridge exactly when there are no edges “out of” the subtree rooted at v.
- How can we compute this “out of” property precisely? Use the squirrel.

Bridge-Finding



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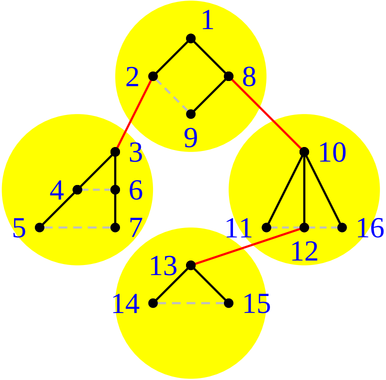

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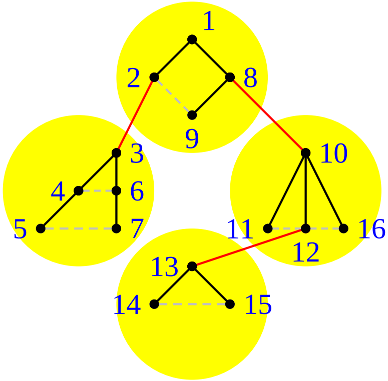

- What does "out of the subtree" mean?
- For each node let $low(v)$ be the minimum of its prelabel, its non-tree neighbours' prelabels, and its children's *low* values.
- For each node let $high(v)$ be the maximum of its prelabel, its non-tree neighbours' prelabels, and its children's *high* values.
- Can show that $(P[v], v)$ is a bridge if and only if $low(v) = pre(v)$ and $high(v) = pre(v) + subtreesize(v) - 1$.

Bridge-Finding

- An *articulation point*, analogous to a bridge, is a *vertex* whose deletion causes a graph to be disconnected.
- By refining the ideas above we can get a $O(n + m)$ time algorithm for articulation points. The formulation is cleanest using DFS because then there are no *cross edges* (edges uv such that neither u nor v is an ancestor of the other).
- Note that the naive algorithm for articulation points — delete each point in turn and see if the graph is connected — takes $O(n(m + n))$ time.
- You can also compute some other things called *biconnected components* and *blocks*. Roughly speaking, you can cut the graph into parts such that each part can tolerate any single node or vertex failure.

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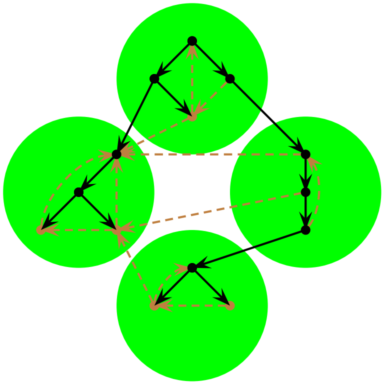
```
isExplored := boolean[v]                          ▷ Initialized to false.
postList := list⟨ int ⟩                           ▷ Initially empty.
procedure StrongComponents(G, v)
    for i := 0 to v − 1 do
        DFS-Order(G, i)
    newOrder := postList.copy().reverse()
    fill(isExplore, false)
    for i in newOrder do
        if !isExplored[i] then
            DFS-Label(G^T, i)                     ▷ When j is explored, label[j] := i.
    return labels
```

Strong Components



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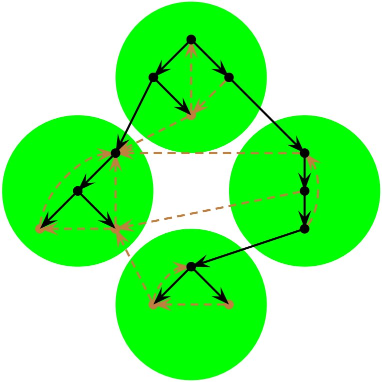

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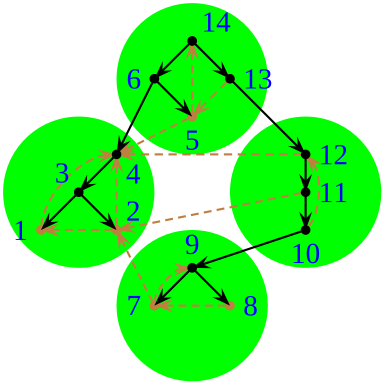

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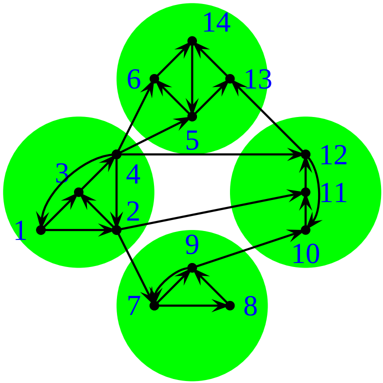

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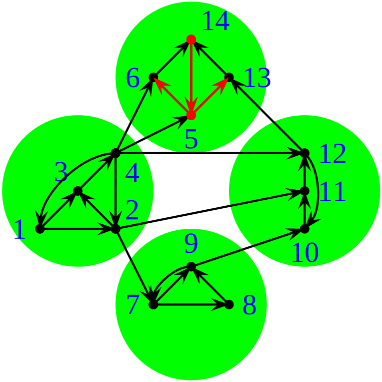

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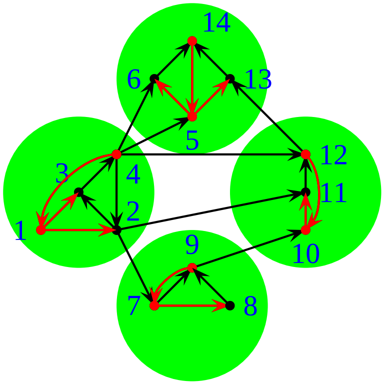

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- The strong component blobs form a DAG (directed acyclic graph).
- Compute postorder, reverse G.
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- Explore other blobs in turn.

Summary

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- DFS can be coded very quickly. Gives many $O(m+n)$ time algorithms: topological sort, biconnectivity, strong connectivity, *planarity*, *triconnectivity*, …

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- `return 0`