UW Math Circle November 29, 2006

1 Combinatorics: The Art of Counting

You have n distinct objects that you want to arrange in n distinct positions. (Each object is to occupy exactly one position and each position is to be occupied by exactly one object.) How many ways are there of doing this?

For example, let's take $n = 3$ and rearrange the letters A, B, C in a sequence so as to make a 3-letter "word." (Generally, we'll use "word" to mean any sequence of letters, whether or not it is actually a word in English or any other language.) There are 6 ways to do this:

$$ABC, ACB, BAC, BCA, CAB, CBA.$$

The total number of ways of arranging n objects like this is called n *factorial* and is denoted $n!$.

$$n! = n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot \dots \cdot 2 \cdot 1.$$

Indeed in the above example we 6 arrangements and $3! = 3 \cdot 2 \cdot 1 = 6$. We also define $0! = 1$ for reasons that will become clear later.

Exercise 1. Prove the above formula.

How many ways are there to arrange the letters $NINE$? Well, by the above logic, there are $4! = 24$ ways to do this, but actually there are only 12:

$$EINN, ENIN, ENNI, IENN, INEN, INNE, NEIN, NENI, NIEN, NINE, NNEI, NNIE.$$

What went wrong? If we treated the Ns as distinct, (say by painting one of them red and the other one blue) then indeed 24 would be correct, but once we become colour-blind, every word in the list appears exactly twice (i.e., for an uncoloured word, there are two possible ways to paint the Ns). So the number x of words in our list should satisfy $2 \cdot x = 4!$, in other words, $x = 12$ as listed above.

Exercise 2. Generalize the above argument. For example, calculate the number of different words that can be obtained by rearranging the letters in $CANADA$, $CIRCLE$, and $MATHEMATICS$.

2 Binomial Coefficients

The version of the previous problem where the word only has two distinct letters gives rise a very special object. The *binomial coefficient* of x and y is denoted $\binom{x}{y}$ and defined by

$$\binom{x}{y} = \frac{x!}{y!(x-y)!}.$$

Let's say we have m A's and n B's to rearrange. Then the argument form the previous section tells us that the number x of ways to rearrange them satisfies $x \cdot m! \cdot n! = (m+n)!$, which we can solve for x and rewrite as $x = \binom{m+n}{m}$.

Exercise 3. An alternative definition is to say that $\binom{x}{y}$ is the number of ways of choosing a set of y items out x. (Here the order in which we pick the numbers does not matter.) Prove this definition is equivalent to the one above.

For this reason $\binom{x}{y}$ is sometimes called x *choose* y.

Exercise 4. Prove the following identities:

- $\binom{m}{n} = \binom{m}{n-m}$, for all integers $m \geq n \geq 0$.
- $\binom{m}{0} = \binom{m}{m} = 1$, for all integers m.
- $\binom{m}{n} = \binom{m-1}{n} + \binom{m-1}{n-1}$, for all integers $m \geq n \geq 0$. *(This is "Pascal's Identity".)*

A deck of playing cards (which I assume you are familiar with) consists of 52 cards, broken down into 4 suits and 13 ranks ($52 = 13 \times 4$). A *poker hand* consists of any 5 cards from the deck.

Exercise 5. How many total poker hands are there (answer : $\binom{52}{5}$)? How many total poker hands are there where all 5 cards are spades (*a spade flush*) (answer: $\binom{13}{5}$)? How many total poker hands are there where you have 3 cards of one rank and 2 cards of another rank (*a full house*)?

Exercise 6. Suppose that the mailman on your street has k envelopes, all identical. If there are n mailboxes, show that the total number of ways to deliver all of the envelopes is $\binom{n+k-1}{k}$. (You can do this with a variant of the word rearranging problem.) What if the envelopes are all distinct?

Exercise 7. What does the first part of the previous question have to do with nonnegative integer solutions of the equation

$$x_1 + x_2 + \cdots + x_n = k?$$

3 Probability

Suppose that we perform some sort of "experiment" that has a finite number of "outcomes." For example, the experiment could be to shuffle a deck of cards and deal 5 cards, to form a poker hand. The *probability* of an event $\mathcal{E}$ is defined as

$$Pr[\mathcal{E}] = \frac{\text{the number of outcomes where } \mathcal{E} \text{ happens}}{\text{the total number of possible outcomes}}.$$

This definition is wrong in a couple of ways, but if you have never seen probability before, it serves as a good introduction, and I will refine the definition as we go along.

For example, the probability that all 5 of the cards are spades is $\binom{13}{5}/\binom{52}{5}$. Just to look at in a slightly different way, let's rewrite it:

$$\frac{\binom{13}{5}}{\binom{52}{5}} = \frac{13!/(5!\cdot 8!)}{52!/(5!\cdot 47!)} = \frac{13!\cdot 47!}{8!\cdot 52!} = \frac{(13\cdot 12\cdot \cdots \cdot 1)\cdot(47\cdot 46\cdot\cdots\cdot 1)}{(8\cdot 7\cdot\cdots\cdot 1)\cdot(52\cdot 51\cdot\cdots\cdot 1)} = \frac{13}{52}\frac{12}{51}\frac{11}{50}\frac{10}{49}\frac{9}{48}.$$

The expression on the right could be obtained in a different way, as follows. The probability that the first card dealt is spades is $13/52$, since 13 of the 52 cards are spades. If the first card is not a spade then we stop, otherwise, there are 12 spades remaining out of a total of 51 cards, so the probability that the second card is spades is $12/51$. Similarly, if the first two cards are spades, the third one is spades with probability $11/50$, etc.

Property 1. *If p is the probability of some event, then* $0 \le p \le 1$.

(In quantum mechanics, probabilities are complex numbers with magnitude between 0 and 1, but that is far beyond our scope today.)

Property 2. *For any event $\mathcal{E}$, we have*

$$Pr[\mathcal{E} \textit{ does not occur}] = 1 - Pr[\mathcal{E}] \qquad \textit{or in other words} \qquad Pr[\mathcal{E}] + Pr[\mathcal{E} \textit{ does not occur }] = 1.$$

More generally, if events $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_k$ are such that exactly one *of the $\mathcal{E}_i$'s always occur, then*

$$Pr[\mathcal{E}_1] + \Pr[\mathcal{E}_2] + \cdots + \Pr[\mathcal{E}_k] = 1.$$

We say that two events $\mathcal{E}$ and $\mathcal{F}$ are *independent* if

$$Pr[\mathcal{E} \text{ and } \mathcal{F}] = Pr[\mathcal{E}] \cdot Pr[\mathcal{F}].$$

Informally, $\mathcal{E}$ and $\mathcal{F}$ are independent if the outcome of one has no effect on the outcome of the other. As another example, let's consider rolling a die. (A die is a cube with 6 sides, and the sides have the numbers from 1 to 6 printed on them.) When you roll one die, every number $1, 2, \ldots, 6$ has probability $1/6$ of showing up on top. The *value* of the roll is the number on top. It is usually safe to assume that when you roll two dice, they are independent; so for example

$$Pr[\text{3 on top of 1st die and 4 on top of 2nd die}] = Pr[\text{3 on top of 1st die}]\cdot Pr[\text{4 on top of 2nd die}] = 1/6\cdot 1/6 = 1/36.$$

By making a diagram (or using some combinatorics), for each integer k, you can determine the probability that the sum of the two dice is equal to k.

Exercise 8. Roll two dice and add up their values. What number is most likely to occur?

The following two questions can be solved with combinatorics, but there are also very pretty and short proofs that avoid any complicated math. (Hint: both questions have the same answer.)

Exercise 9. You flip a coin $n \geq 1$ times. What is the probability that the number of heads you obtain is even?

Exercise 10. Amelia flips n coins and Bruce flips $n+1$ coins. What is the probability that Bruce gets more heads than Amelia?

3.1 Unfair Coins

In gambling, some unscrupulous people have been known to cheat by manipulating the dice, coins, or cards in such a way that they are biased to come up with some numbers more often than others. For example, while a *fair coin* has an equal 1/2 probability of coming up heads or tails, an *unfair coin* could be more likely to come up heads than tails. We have to revise our definition of probability to account for this.

Definition 1. *Let $\mathcal{E}$ be an event in some experiment. Suppose we perform the experiment n times, and let $observed(n)$ denote*

$$observed(n) = \frac{\textit{the number of trials out of the first } n \textit{ where } \mathcal{E} \textit{ occured}}{n}.$$

Then $Pr[\mathcal{E}]$ is the limit of $observed(n)$ as n goes to infinity.

So an unfair coin, say with probability $h = 2/3$ of coming up heads and probability $t = 1 - h$ of coming up tails, in the long run, will show about twice as many heads as tails. We'll always use h to denote the probability that an unfair coin comes up heads, and $t = 1 - h$ to denote the probability that it comes up tails.

Exercise 11. What is the probability that our unfair coin produces k heads in n tosses? (answer: $\binom{n}{k} h^k t^{n-k}$)

Fix an integer n and let $\mathcal{E}_i$ denote the probability that we get exactly i heads. Since exactly one of the events $\mathcal{E}_0, \mathcal{E}_1, \ldots, \mathcal{E}_n$ always happens, we have that

$$1 = \sum_{k=0}^{n} Pr[\mathcal{E}_k] = \sum_{k=0}^{n} \binom{n}{k} h^k t^{n-k}.$$

This is a special case of the *binomial theorem*, which states that for any x and y,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

(To see the correspondence, take $x = h, y = t$ and note $(x+y)^n = (h+t)^n = 1^n = 1$.)

3.2 The Birthday "Paradox"

Let's assume that in a room there are n people, and the birthdays of those people are independent random days of the year. What is the probability that two people have the same birthday? This involves some messy calculations so it is usually rephrased in the following way: I offer to pay you \$1 if there are no two people in the room with the same birthday, but you must pay me \$1 if two such people exist. It is not hard to see that you should accept the offer if and only if the probability of two people having the same birthday is at most 1/2.

There are 366 possible days in a year, so one might accept the bet if there are less than 366/2=183 people. However, this is a bad idea — even if there are only 23 people, the probability that you will win the bet is

$$\frac{366}{366}\frac{365}{366}\frac{364}{366}\cdots\frac{344}{366} \approx 0.49368.$$

One way to get some intutition about it is that, in n people, there are $n(n-1)/2 = \binom{n}{2} \approx n^2/2$ pairs of people, and the probability that any of these pairs overlap is 1/366, which suggests the threshold is $n \approx \sqrt{2 \cdot 366} \approx 27$. (This approach can't be formalized, though.)

4 The Principle of Inclusion-Exclusion (PIE)

This section is best explained with *Venn diagrams* which I have not included in the notes, only the formulas are given below. Here $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \ldots, \mathcal{E}_n$ are some events.

$$Pr[\mathcal{E}_1 \text{ or } \mathcal{E}_2] = Pr[\mathcal{E}_1] + Pr[\mathcal{E}_2] - Pr[\mathcal{E}_1 \text{ and } \mathcal{E}_2].$$

$$Pr[\mathcal{E}_1 \text{ or } \mathcal{E}_2 \text{ or } \mathcal{E}_3] = Pr[\mathcal{E}_1]+Pr[\mathcal{E}_2]+Pr[\mathcal{E}_3]-Pr[\mathcal{E}_1 \text{ and } \mathcal{E}_2]-Pr[\mathcal{E}_1 \text{ and } \mathcal{E}_3]-Pr[\mathcal{E}_2 \text{ and } \mathcal{E}_3]+Pr[\mathcal{E}_1 \text{ and } \mathcal{E}_2 \text{ and } \mathcal{E}_3].$$

$$Pr[\mathcal{E}_1 \text{ or } \mathcal{E}_2 \text{ or} \cdots \text{or } \mathcal{E}_n] = \sum_{S \subseteq \{1,2,\ldots,n\}} (-1)^{|S|-1} Pr[\mathcal{E}_i \text{ happens for all } i \text{ in } S].$$

Remember the connection between probability and combinatorics? We can also write the above formulas in terms of set sizes, e.g., $|A \cup B| = |A| + |B| - |A \cap B|$ and more generally

$$\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{S \subseteq \{1,2,\ldots,n\}} (-1)^{|S|-1} \left| \bigcap_{i \in S} A_i \right|.$$

Exercise 12. How many numbers between 1 and 1000 are not divisible by 2, 3, or 5?

Exercise 13. What is the probability that a poker hand contains at least one card from each suit? (Express your answer using a small number of binomial coefficients.)

Exercise 14. A drunken mailman has n houses on his route, and one envelope addressed for each house. He delivers all of the envelopes at random (i.e., you may consider that he picks one of the $n!$ possible assignments at random). What is the probability that no house gets the right envelope?

In the previous problem (the "Bernoulli-Euler problem of the misaddressed letters"), the limiting probability as n goes to infinity can be determined using the *exponential series formula:*

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

5 More Interesting Problems

Exercise 15. ("Let's make a Deal") We play the following game: I have three boxes, and inside one of the boxes, I hide a \$1000 bill. I arrange the boxes randomly you take one. After you do this, of the two unpicked boxes, I open up an empty one. Then I offer you the chance to switch the box you took with the remaining closed box. You want to end up with the box containing the bill. Should you switch when I make the offer?

Exercise 16. There are 2^n teams who play a tennis tournament. When a team loses a match, it leaves the tournament. In each round, all of the remaining teams are paired up randomly and the pairs play a match; assume that the result of any given match is a purely (independent) random 50-50 outcome. What is the probability that my team ever plays your team?

Exercise 17. There is an infinite line of cells numbered 1, 2, 3, ..., and you start out in cell 1. At each turn you flip a coin; if it is heads, you advance by one cell, and if it is tails, you jump over the next cell and hence advance by two cells. What is the probability that at some point you land in cell 10? Generalize this to cell n.

Exercise 18. You have n^2 coins, n of which are silver. Suppose you arrange the coins at random, in an $n \times n$ grid. Prove that the chance that at least one row has no silver coin is

$$1 - \frac{(n-1)!(n^2-n)!n^{n-1}}{(n^2-1)!}.$$

For Next Week:

Exercise 19. Each time you buy a box of Cheeri-Omegas cereal, you win a prize! There are n types of prizes, and the cereal company puts a random type of prize into each box. How many boxes of cereal will you need to buy before you expect to have at least one prize of each type?