UW Math Circle Problems for October 3 & 10 '07

Some Problems on Fibonacci Numbers

We define the first two Fibonacci numbers by $F_1 = 1$ and $F_2 = 1$. For $k \geq 3$, we define the kth Fibonacci number by $F_k = F_{k-1} + F_{k-2}$.

- Show that $F_{k-1} \times F_{k+1} = F_k^2 + (-1)^{k-1}$ for all $k > 1$.
- Find a simple formula for $F_1^2 + F_2^2 + \cdots + F_k^2$ and prove your answer. (Hint: try subtracting F_{2k-1}; or you can try subtracting F_{2k}.)
- (harder) Show that F_k divides F_m whenever k divides m.
- Show that F_{k-1} and F_k have no common divisors other than $+1$ and -1 (easy if you know the Euclidean algorithm for finding GCDs).

We define the first two Lucas numbers by $F_1 = 2$ and $F_2 = 1$. For $k \geq 3$, we define the kth Lucas number by $F_k = F_{k-1} + F_{k-2}$.

- Define $Q'_k = L_{k+1}/L_k$ for all $k \geq 1$. What can you say about Q'_k compared to Q_k (recall $Q_k = F_{k+1}/F_k$)?

We also talked about packing dominoes and small squares into rectangles ("puzzles"). Although this doesn't have to do with Fibonacci numbers, can you figure out a recurrence relation to count the number of ways to fill a $2 \times n$ rectangle with 2×1 squares (dominoes) and 1×1 squares?

Sequence Problems

- Let $f(n)$ denote the sum of the first n terms of the sequence $0, 1, 1, 2, 2, 3, 3, 4, 4, \ldots$. Show that when s and t are positive integers and $s > t$, that $f(s+t) - f(s-t) = s \times t$.
- Can you find a formula for $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + n \cdot (n+1) \cdot (n+2)$?
- Can you find a formula for $1^2 + 3^2 + \cdots + (2n+1)^2$?
- Can you find a formula for $1/(1 \cdot 2 \cdot 3) + 1/(2 \cdot 3 \cdot 4) + \cdots + 1/(n \cdot (n+1) \cdot (n+2))$?
- Let $a_1 = 2$ and $a_k = a_{k-1}^2 - a_{k-1} + 1$ for $k \geq 2$. Show that all of the numbers a_k are relatively prime to each other. (The first few terms are $2, 3, 7, 43, 1807, \ldots$)

Polynomial Problems

- Let r_1, r_2, r_3 denote the roots of $x^3 - 2x^2 = (x+3)^2$. What is $r_1r_2 + r_2r_3 + r_3r_1$?
- Let $f(x) = x^4 + x^3 + x^2 + x + 1$. What is the remainder when $f(x^5)$ is divided by $f(x)$?
- Let $g(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$ where $a_0, a_1, \ldots, a_n$ are integers and $a_0 \neq 0$. Suppose that distinct integers a, b, c, d exist so that $g(a) = g(b) = g(c) = g(d) = 5$. Show that $g(8) \neq 8$.
- Let $p(x)$ be some polynomial of degree n so that $P(k) = 1/k$ for $k = 1, \ldots, n+1$. Determine $P(n+2)$.
- (hard if you haven't seen it) Factor $x^4 + 4 = 0$.
- (USAMO 1984, hard) Suppose that $Q(x)$ is a polynomial of degree $3n$ so that

$$P(0) = P(3) = \cdots = P(3n) = 2$$

$$P(1) = P(4) = \cdots = P(3n-2) = 1$$

$$P(2) = P(5) = \cdots = P(3n-1) = 0.$$

If $P(3n+1) = 730$, find n.