What is a game? A game is any situation where two or more players make choices — which may be simultaneous, sequential, or a combination of both — and get a utility at the end of the game, depending on the choices made.

Axioms of Mathematical Game Theory

1- every player is greedy and one-minded: they only want to maximize their own utility
2- every player knows all rules of the game, all legal moves, and the definition of everyone's utility
3- every player is perfectly rational (makes no mistakes)
4- every player knows 1, 2, 3, 4 are true

What is the purpose of mathematical game theory? To determine good strategies and predict outcomes of games. But in general, no "predictable" outcome or "best" strategy exists; for example:

Bach or Stravinsky. Two players want to see a concert and they need to decide if they are going to go to a Bach concert or a Stravinksky concert. Both players simultaneously pick B or S. If both players pick B, player 1 gets 100 points and player 2 gets 50 points. If both players pick S, player 1 gets 50 points and player 2 gets 100 points. If the players say different letters, both get 0 points (they stay home).

We forbid players from discussing before the game — they are given the rules of the game and both immediately have to make a decision.

Despite the above point, here are some **examples of games with only one logical outcome**.

The Pollution Game. Some number n of companies each need to decide whether to "go green." They all simultaneously decide G (green) or P (pollute). Let k be the number of companies that chose P. Each company that chose G has to pay \$(3+2k), and each company that chose P has to pay \$2k.

The 2/3-Average Game. Some number n of players each simultaneously choose a number from 0 to 100. Then, we compute the average a of all choices. The player whose choice is closest to $2a/3$ wins, and all other players lose.

Election. Two players simultaneously pick "political positions" in the real interval [0, 1] (or the unit circle / sphere / cube). We determine the area within the interval / circle / sphere / cube that is closest to each of the two players; we think of each player "owning" that area. If one player owns more area than the other, they win and the other loses. If the players pick the same position, or if they own the same amount of area, they tie.

The Centipede Game is a two-player game. In each round (1, 2, …) one of the players is given the choice to either stop or continue the game. The game ends as soon as someone chooses to stop. Player 1 decides in the first round, then player 2 in the second round, player 1 again in the third round, etc. alternating until the game stops. If the decision to stop was made in the kth round, then the player who chose stop gets k+2 dollars, and the other player gets k dollars. In the 100$^{\text{th}}$ round, we force the game to end by specifying that player 2 can only choose stop.

Democratic Pirates. Pirates #1, #2, …, #10 have found 100 gold coins. To decide how to share the coins, first pirate #1 (the fiercest) proposes a division of the 100 coins amongst the 10 pirates; then all ten pirates vote to accept or reject the proposal. If more than half of the pirates reject, they throw pirate #1 overboard, and pirate #2 gets to propose a division amongst the remaining 9 pirates; then the remaining nine pirates vote either to accept or throw pirate #2 overboard, etc. Each pirate prefers to receive 0 coins than to be thrown overboard.

Second-price Auction. Some number n of players are bidding on an item. Each player i values the item at \$$v_i$; and each player i only knows v_i, and not v_j for any other player j. (So in this case, Axiom 2 does is not entirely true.) Each player i simultaneously chooses a bid b_i. Let i^* be the player who bid the highest (breaking ties arbitrarily), and j^* be the player who bid second highest. Then we give player i^* the item and charge them b_{j^*} — and this means that player i's utility is v_{i^*}-b_{j^*}, and every other player's utility is 0.

A neat fact about the last game (second-price auction) is that there is a provable unique predictable outcome even though Axiom 2 is violated. A more striking fact is that a first-price auction — the same but except that we charge player i^* their own bid, so they get utility v_{i^*}-b_{i^*} — does not have a unique predictable outcome.

Sometimes an interesting puzzle can come about just in designing a good game. If two players are going to share a cake, there is an ancient protocol which ensures that it can be divided fairly between the two players: you cut, I choose. (I.e., player 1 cuts the cake into two pieces; then player 2 chooses one of the two pieces to take; then player 1 keeps the unchosen piece.) What is a fair protocol for 3, 4, or more players? (Bonus: what if the players don't value pieces equally, e.g. Alice likes sprinkles, Bob likes frosting, and Charlie likes chocolate?)

When there are only two players, the Pollution Game is an example of a classic situation called the Prisoner's Dilemma. The only logical outcome is for both players to choose P, even though they would both spend less if they had both picked G. In society, however, we often observe that people in this sort of situation will pick G if they have reason to believe that they will have to work again with that player in the long run. In fact, in an "infinitely repeated prisoner's dilemma," you can prove that everyone always picking G is actually a logical outcome.

In the wild west, it was common for cowfolk to settle their differences with a n-uel (when n=2, it was called a duel). Each player has an infinite supply of bullets. We assume that the players are standing in a circle and that one of the players has been chosen to go first. Each player has a fixed *marksmanship* which is a real number between 0 and 1. In each round, the current player fires one bullet; if a player is shot at, they die in that round with probability equal to the marksmanship of the shooter. Then, play passes to the clockwise-next player who is not yet dead. Play stops if only one player is alive. Suppose there are three players; the first player has marksmanship 1/3, the second has marksmanship ½, and the third has marksmanship 1. What should player 1 do on his first turn?

The following well-known game still has a unique predictable outcome, but it is quite different from all the previous games mentioned.

<u>*Rock-Paper-Scissors*</u>. Two players each simultaneously pick rock, paper, or scissors. If the players make the same choice, they tie. Otherwise, rock wins over scissors; scissors wins over paper; and paper wins over rock.

The rest of the handout deals with **combinatorial games**, defined as follows:

1) There are *two players* who play *alternately*
2) There are several, but finitely many, game positions, we will call *positions*
3) There is *complete information* (both players know all *options* from each position)
4) Play is *deterministic* (there is no randomness)
5) The game must terminate after a *finite* number of moves

For example,

<u>*The Ten-coin game.*</u> There is a pile of ten coins. On your turn, you may remove 1 or 2 coins from the pile. The person to take the last coin wins.

In principle, the following analysis allows you to play perfectly in any combinatorial game.

Rules for W-L position analysis:

1) A game-winning move goes to a W position.
2) A game-losing move goes to an L position.
3) If a position has *any* options marked W, mark it L
4) If a position has *only* options marked L, mark it W

(by adding a couple more rules you can extend this to also work for games that have draws)

<u>Wythoff's Game</u>
Setup: There are two piles, respectively containing *M* and *N* coins.
Rules: On your turn, remove any number of coins from any one pile, or an equal number of coins from both piles.
Winning: The person to take the last coin wins. [*The W and L positions have a nice pattern.*]

<u>Kayles</u> (example of Copycat Principle — the first player has a winning strategy)
Setup: There is a line of *N* coins, one at each location 1, 2, 3, …, *N*.
Rules: On your turn, remove any 1 coin, or a 2 coins at locations *X*, *X*+1 (i.e., adjacent coins).
Winning: The person to take the last coin wins.

<u>Knight Rider</u> (example of a Pairing Strategy — the second player has a winning strategy)
Setup: The game takes place on a 8x8 chessboard, using one knight. (They move in 2x1 L patterns.)
Rules: The first player chooses the knight's initial position, then the second player moves it, then the first player moves it, etc. The only rule is that the knight can never occupy the same square twice.
Winning: The last person to move the knight wins.

<u>Triples and Singles</u> (example of a Parity Argument)
Setup: Start with a heap of *N* counters.
Rules: On your turn, split any heap into two non-empty heaps. A heap of 3 may not be split.
Winning: The last person to make a move wins.

<u>Chomp</u> (example of Strategy Stealing — the second player does not have a winning strategy)
Setup: Start with a rectangular chocolate bar.
Rules: On your turn, eat any remaining square, and all squares above-and-or-to-the-right of it.
Winning: The player to eat the (poisoned) lower left square loses.

<u>Nim</u>
Setup: There are several piles containing several coins each. Traditionally, start with {3, 5, 7}.
Rules: On your turn, you may remove any number of coins from any one pile.
Winning: The person to take the last coin wins.

The Sprague-Grundy Theorem for Impartial Games (Sprague 1936, Grundy 1939)

All positions in an impartial combinatorial game are equivalent to a pile of Nim counters. We call the number of counters in this equivalent pile that position's *Nim value*. When playing such a game, you can guarantee a win if you are ever able to bring the game to a position with value 0.

Calculating Nim Values – Parallel Games
Suppose you are given two games to be played in parallel – on your turn you may move in either one of the two games, and the last person to make a move in either game wins. If the Nim values of the initial positions for the two games are A and B, then the value of the combined game is A xor B (write both values in binary, add without carrying, and convert back from binary to get the resulting value).

Calculating Nim Values – The Mex Rule (mex = *m*inimum *ex*cluded)
Suppose that we want to calculate the Nim value for a particular position P in an impartial combinatorial game, and its options have Nim values $V_1, V_2, V_3, \ldots V_N$. Then the Nim value of P is

$\text{mex}(V_1, V_2, V_3, \ldots V_N)$ == the least non-negative integer not equal to V_i for any i.

<u>Treblecross</u> a.k.a. 1D Tic-Tac-Toe (you can win by using the Mex and Xor rules)
Setup: There is a strip of paper with *N* empty boxes on it.
Rules: On your turn, you mark an **X** in any box.
Winning: The first person to complete a sequence of three adjacent marked boxes wins.

<u>Grundy's Game</u> (you can win by using the Mex and Xor rules)
Setup: There is a pile containing *N* coins.
Rules: On your turn, you may take any pile and break it into two nonempty, unequal-sized piles.
Winning: The last person to make a move wins.

<u>The Cube</u>
Setup: You are given a cube, initially all of its edges are unpainted.
Rules: On your turn, paint any unpainted edge red, blue, or green, subject to the following rule: if two edges meet at a vertex, they are not allowed to be painted the same colour.
Winning: The last person to make a move wins.

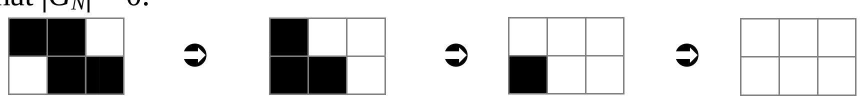
Some other interesting problems
There is a classroom of 25 students, with their seats arranged in a 5x5 grid. Each student is assigned to the same desk every day. A superstitious teacher wants all of the students to change their positions in such a way that *the new desk for each student is adjacent (up, down, left or right) to their old desk.* Why can't the students all do this?

Consider N people outside, each armed with a Super Soaker water gun. Assume that for every three people i, j, k, the distance from i to j is not equal to the distance from j to k. Now we tell everybody to shoot the person that is closest to them. If N is odd, prove that someone does not get shot.

You are given an infinite chessboard with N black squares, and the rest are white. Let the collection of black squares be denoted G_0. In each iteration, t=1,2,3,… we recolour some squares in the following way: for each square s such that both its neighbour above and its neighbour to the right have a different colour than s, we change the colour of s. The new configuration of black squares is called G_t, t=1,2,3,… .

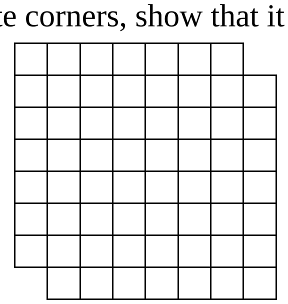
a) Show that it is possible for $|G_1|$ to be larger than $|G_0|$.
b) [*hard*] Show that $|G_N| = 0$.

Here is an example:

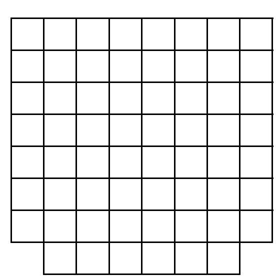


Suppose we cut two corners off of an 8x8 chessboard. There are 62 squares left, so it appears that they can be covered by 31 2x1 rectangles. (You can put a 2x1 rectangle in two orientations, either covering two left-right adjacent squares or 2 up-down adjacent squares). Show that you can indeed cover the chessboard if the corners were taken from adjacent corners, but if the corners were taken from opposite corners, show that it is impossible.

Impossible:



Possible:



e.g. on 4x4 board: