Max-Weight Integral Multicommodity Flow in Spiders & High-Capacity Trees

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Maximum–weight integer multicommodity flow

- **Input**: graph with edge capacities c_e; pairs of terminals, profit w_i for each commodity i. **Goal**: integral y_i-flows respecting capacities,

$$\text{such that} \quad \sum_i w_i y_i \quad \text{is maximized}$$

- E.g. transport beer kegs between brewer and customer along roads
- (Max–profit *fractional* multiflow can be found in poly–time via LP)

Bad News, Good News

- Hard to $\log^{1/3-\epsilon} n$-approximate [AZ 05]
  - $O(\log^{1/2} n)$ approx alg by randomized rounding
  - If all edge capacities $\geq \mu$, $1+O(\log n/\mu)^{1/2}$-apx
- Well-studied special case: graph is a tree
  - Same as *weighted path packing*
  - Poly-time solvable for stars, paths, unit-capacity trees but APX-hard in general [GVY 93]
  - 2-approximation for unit weights [GVY 93]
  - 4-approximation for general weights [CMS 03]
  - [CJR 99] studied naïve LP, e.g. integrality gap

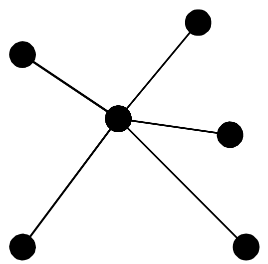
Our Results

- Exact solution when tree is a *spider*
  - spider: subdivision of star
- $(1+6/\mu)$–Approximation Algorithm

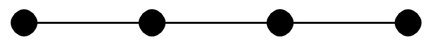
  μ: minimum edge capacity
  - Iterated LP relaxation yields additive guarantee
    - Answers an open question of [CMS 03]
  - Tweak into a multiplicative guarantee
    - Uses known techniques: scaling, iterated rounding
    - Uses new techniques: LP structure lemma, iterated relaxation of auxilliary covering problem
  - Matching $(1+\epsilon/\mu)$–hardness result
  - For $\mu \geq 2$ we get 3–approx via [CJR 99]

Exactly Solvable Cases

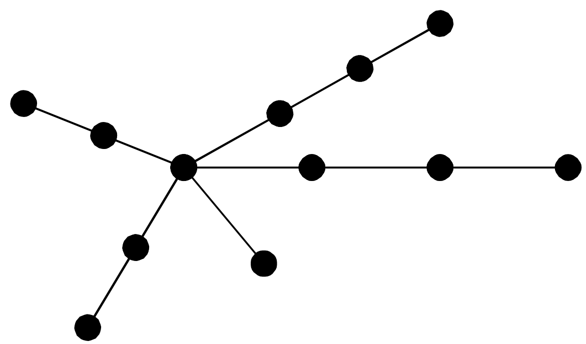
- In *X*, integer multiflow reduces to Y:
- *X* = stars, *Y* = *b*–matching
- *X* = paths, *Y* = max–cost circulation
- *X* = cap–1 trees, *Y* = DP + matching
- *New:* *X* = spiders, *Y* = bidirected flows (yields LP characterization)



A 5-tip star



A path

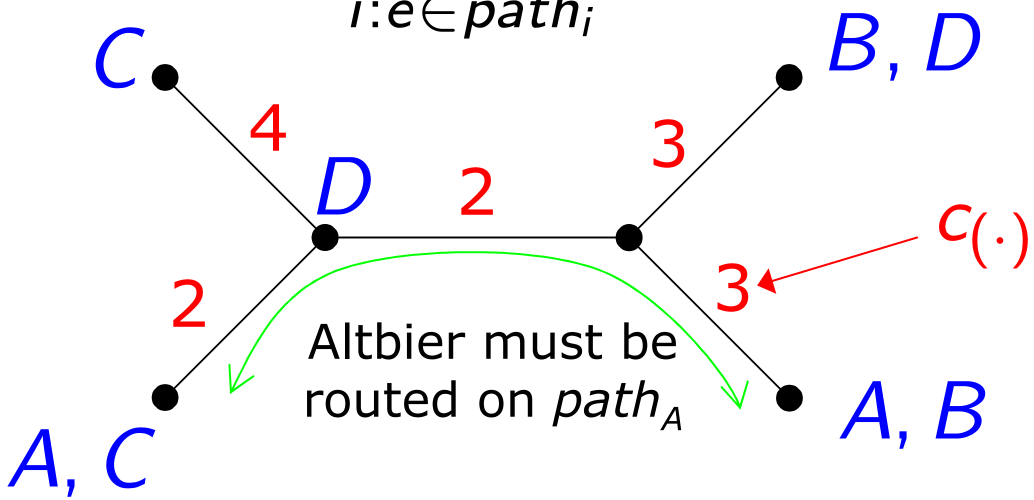


A 5-legged spider

Notation/Formulation

- $path_i$: tree path for commodity i's terminals
- feasible flow: nonnegative integral y with

$$\forall e : \sum_{i:e\in path_i} y_i \leq c_e$$



- E.g. $y_A = 1$, $y_B = 2$, $y_C = 1$, $y_D = 1$ is feasible

Relax!

- Drop integrality $\Rightarrow$ naïve LP relaxation

$$\max w \cdot y : \quad y \geq 0, \quad \forall e \sum_{i:e \in path_i} y_i \leq c_e$$

- y^{OPT} denotes optimal LP solution
- "LP-based α-approximation algorithm:" produce solution y with $w \cdot y \geq w \cdot y^{OPT} / \alpha$
- Our approach uses *iterated rounding* [Jain 98] & *iterated relaxation* [LNSS 07]

Iterated Relaxation Idea

- Start by routing the integral part of y^{OPT}; replace capacities by residual capacities
- Afterwards, $0 \leq y_i^{OPT} < 1$ WOLOG, for all i
- In each remaining iteration, solve LP and
  - route one more unit of flow and replace capacities by residual capacities,

    or,
  - discard capacity constraint for an edge e

Iterated Relaxation

1. Solve the LP, obtaining y^{OPT}

Stop iterating once y^{OPT} is all-0

2. If $y_i^{OPT} = 1$ for any i:
   - Route 1 unit of i, update capacities
   - Discard i

Decrease in LP value equals increase in output value

3. Else
   - Find e on at most 3 $path_i$'s
   - Delete constraint for e

LP value does not decrease

4. Go back to step 1

Conclusion: output solution has value ≥ initial LP value, but violates capacity constraints by as much as +2.

Why Iterated Relaxation Works

- (LP structure lemma) If y^* is an extreme LP solution and $0 \le y_i^* < 1$ for each commodity i, then some edge e lies on at most three $path_i$'s.
  - Proof idea
  - y^* is unique solution to set of |support(y^*)| linearly independent tight capacity constraints
  - Contract other edges (drop constraints) ⇒ T'
  - Independence, integrality of c_e's ⇒ every degree-2 vertex in T' is a terminal for at least 2 commodities
  - Counting argument ⇒ some leaf in T' is a terminal for at most 3 commodities. Its incident edge works. □

Fixing the +2-violation

- Iterated rounding gives y s.t. $w \cdot y \geq w \cdot y^{OPT}$,

$$\forall e : \sum_{i:e \in path_i} y_i \leq c_e + 2$$

- But… we want a solution y' such that the same holds without the "+2"
  - $w \cdot y'$ should still be large compared to $w \cdot y^{OPT}$
- Approach: find "decrease" z, set $y' := y - z$
  - Finding a cheap integral decrease is special case of [Jain 98]

Fixing the +2-violation

- Define $overload_e := \max\{0, \sum_{i:e\in path_i} y_i - c_e\}$
- z is a feasible decrease if
$$\forall e : \sum_{i:e\in path_i} z_i \geq overload_e$$
- Crucially, [Jain 98] gives 2-approx algorithm relative to the optimal *fractional* solution
- $\check{z} = 2y/(\mu+2)$ is feasible fractional decrease
  - Proof idea: $y-\check{z}$ is smaller than y by a $\mu/(\mu+2)$ factor. Works since overload ≤ 2 & capacity $\geq \mu$.

Overall Algorithm, Analysis

- Let y^{OPT} be an optimal integral flow
- Iterated relaxation gave a +2-violating solution y with $w \cdot y \geq w \cdot y^{OPT}$
- Iterated rounding [Jain 98] yields solution z to auxilliary covering problem with

$$w \cdot z \leq 2 w \cdot \check{z} = 4 w \cdot y / (\mu + 2)$$

- So $w \cdot (y - z) \geq w \cdot y^{*} (1 - 4/(\mu+2))$

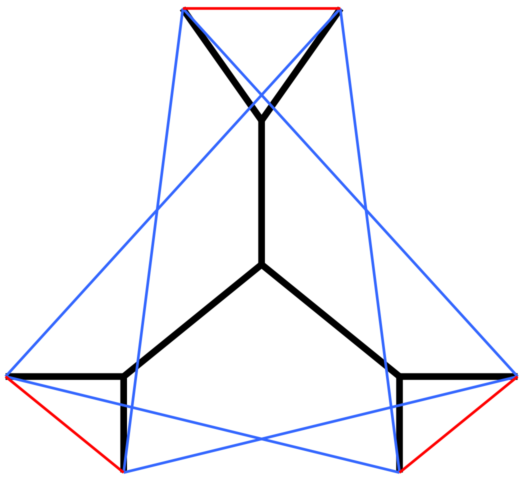
$$\geq w \cdot y^{OPT} / (1 + 4/\mu + 24/(\mu^2 - 6\mu)) \square$$

Loose Ends

- Slightly tighter analysis gives a better ratio for specific values of μ:
  - decreasing overload in two steps instead of one yields $1 + 4/\mu + 6/(\mu^2-\mu)$ approximation
  - for $\mu=2, 3$ use load-halving argument of [CJR 99] to get better approx ratio of 3
- $(1+\epsilon/\mu)$-hardness, for any fixed μ, follows by modifying original APX-hardness proof of [GVY 93]

Can We Do Better?

- Can we find a +1–violating solution, in place of a +2–violating solution?
  - No evidence it's completely impossible…
  - But "3" in structure lemma cannot be made "2":



[CJR 99]

- Capacities equal 1
- Blue: flows of fractional value ¼
- Red: flows of fractional value ½
- Extreme!

Future Work (I)

- We have a 3-approx when all capacities >1 and an exact algorithm when all capacities =1. Can we combine for general instances?
- Adding {0,1/2}-Chvátal-Gomory cuts to naïve LP creates blossom-like inequalities
  - Strengthened LP is *integral* in the case of unit-capacity trees and spiders
  - Can separate them in polynomial time [CF 96]
  - Useful for approximation in general trees?
  - Would hope for a new LP structure lemma

Future Work (II)

- Obtained $1+4/\mu+\ldots$ approx for tree multiflow. Similar existing LP-based results:
  - Demand matching[SV 02]: $1+6^*\text{demand}_{max}/\mu+\ldots$
    - "Demand" multiflow: $y_i \in \{0, \text{demand}_i\}$
  - Smallest *k*-edge-connected subgraph: $1+2/k+\ldots$
- How about these problems?
  - Demand tree multiflow?
    - Best known apx is $1 + O(\text{demand}_{max}/\mu)^{1/2}$
  - *Min-cost k*-edge-connected subgraph?
    - Best known apx is 2

Prosit!