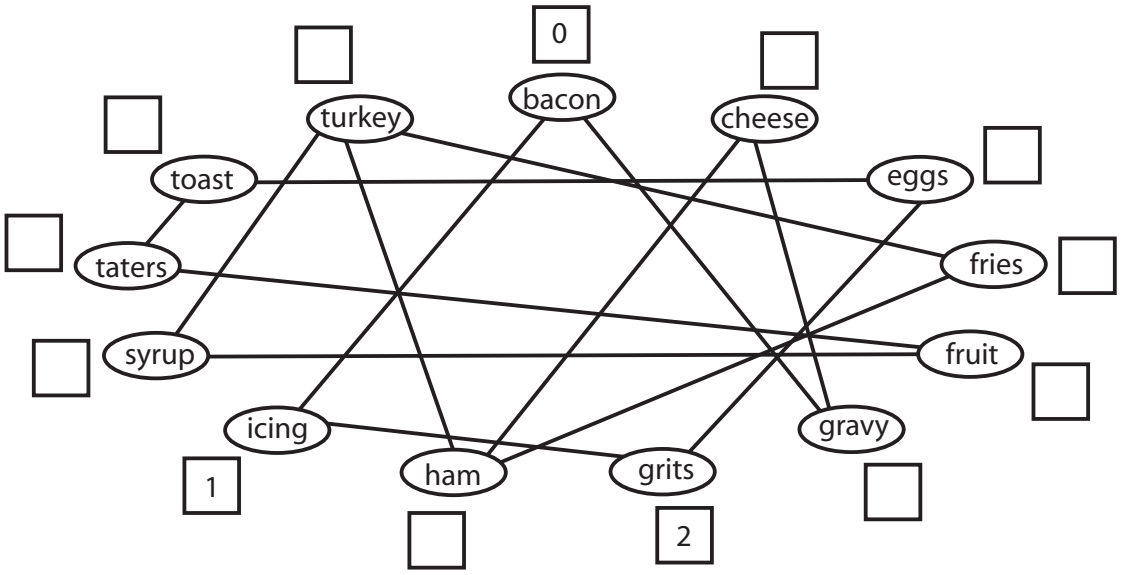
Question 1. In a graph, the *length* of a path is the number of edges it contains, and the *distance* between two vertices u and v is the minimum length of any path from u to v. We want to know if the Bob Evans menu has the *six degrees of bacon* property. In order to do this, compute for each vertex its distance from *bacon* — put your answers in the boxes, note a few are filled in already to get you started.



Question 2. For this problem,

- Create a binary search tree from the letters $KENTUCKY$ and show the final result.
- Delete E and show the result.

Question 3. A drunken ant is walking along the edges of a cube, starting on a corner. Each minute, it randomly picks one of the three corners adjacent to its current position and walks to it.

- What is the probability that after 8 minutes, it is back at the start?
- What is the probability that after 8001 minutes, it is back at the start?

`[should say "gives 1.5"]`

Question 4. In LISP, note that `DIV` is exact division (so for example `(DIV 3 2)` gives 1.5). Define a LISP procedure `DM` which takes two positive integers `X` and `Y` as input, and outputs a two-element list whose first element is the quotient of `X` divided by `Y`, and whose second element is the remainder when `X` is divided by `Y`. For example `(DM 100 7)` should evaluate to `(14 2)`.

```
(DEF (DM X Y)
```

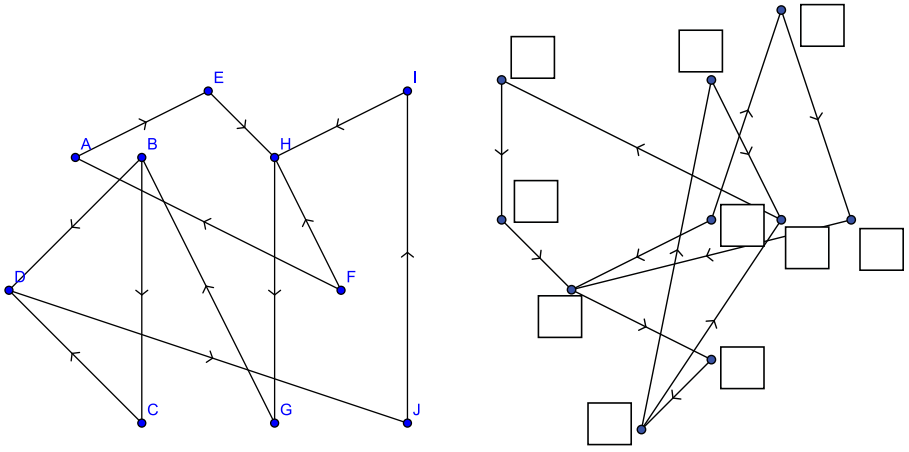
Question 5. Let `DM` be the function defined in the last question. (You do not need to have answered that question correctly to do this one.) If `(EVAL (CONS 'DM (DM X 100)))` evaluates to `(1 1)`, what is the smallest positive integral value `X` could be? What is the largest possible value `X` could be?

Question 6. Simplify the following two expressions as much as possible.

$$(A+\overline{B})(B+\overline{C})(C+\overline{D})(D+\overline{A})$$

$$\overline{B(\overline{A}(B+A)+\overline{C}(B+A))}+C$$

Question 7. Two drawings of the same directed graph are shown below, but the labels from the right drawing have been erased. Draw them back in.



Question 8. Find a value for A and a value for B such that $A\overline{B} + \overline{A}(B(\overline{B} + \overline{A})) \neq \overline{B}(A + B\overline{(AB)})$

Question 9. Find a three-digit decimal number abc_{10} which, when converted to hexadecimal, is the same string backwards, cba_{16}. (We require $a \neq 0, c \neq 0$.)

Question 10. I just opened a textbook whose page numbers are in binary. The left-hand page has an odd number. The two pages facing me altogether show 5 ones and 3 zeroes. What pages am I looking at?

Question 11. Evaluate RSHIFT-3 ((LCIRC-5 (10010110110 AND 01110111100)) OR 10101010101)

Question 12. The *Toffoli gate* has three input wires (I_1, I_2, I_3) and three output wires (O_1, O_2, O_3) defined by the following properties:

- $O_1 = I_1$
- $O_2 = I_2$
- if both I_1 and I_2 are true then $O_3 = I_3$, otherwise $O_3 = \overline{I_3}$

Build a Toffoli gate. (That is to say, insert any standard gates into the picture below and connect everything so the overall circuit meets the specifications above.)

I_1_____ _____O_1

I_2_____ _____O_2

I_3_____ _____O_3

Question 13. The following statements are evaluated one after another. For each statement with a blank next to it, write the value of that expression, or ERROR if an error occurs (at least one statement generates an error).

```
__________ (CONS 'A ())
__________ (CONS () 'A)
__________ (EVAL (CONS (CAR '(ADD 10 4)) (CDR '(1 2 3))))
__________ (MULT (ADD (SUB (DIV 12 4) 8) 10 6) 16)
           (DEF (ABRA F X) (F (F X)))
           (SETQ ABBA '((I LOVE) (LISP ((A LOT)))))
__________ (ABRA CADR ABBA)
           (SET 'A 'A)
           (SET A'B)
__________ A
__________ B
__________ (QUOTE QUOTE)
__________ (EQ 1 '1)
__________ (EQ '1 ''1)
```

Question 14. In this problem we consider FSAs over the alphabet with just a single letter, S. Consider the following regular expression:

$$S(SS)^* \cup SS(SSS)^*(SS \cup S)$$

Write a simplified regular expression which generates exactly those strings which are *not* generated by the one above.

Question 15. Convert 0.2_{10} to binary.

Question 16. Consider the prefix expression $a\,b\,c\,+\,/$ and the postfix expression $/\,+\,c\,b\,a$ obtained by reversing it. (Assume / is exact division, not integer division.)

- What is the value of the prefix expression if $a = 2, b = 3, c = 4$?
- What is the value of the postfix expression if $a = 2, b = 3, c = 4$?
- If $a = 5$ and $b = 2$, and both expressions have the same value, what are the possible values of c?

Question 17. Suppose you create a maximal heap (i.e. letters later in the alphabet have higher priority) from the letters $THISSENTENCEHASTHIRTYONELETTERS$ (note, it does actually have 31 letters).

- Show the heap after the first 5 letters have been inserted.
- What is the internal path length after all 31 letters are inserted?
- What is the external path length after all 31 letters are inserted?
- After all 31 letters are inserted you perform 5 pops. What are the letters popped off?

Hint: you don't need to show the full tree.

Question 18. Consider a version of ACSL Assembly augmented with a queue. The operation `PUSH` pushes the value in the accumulator onto the queue, and the operation `POP` pops a value from the queue and puts this value into the accumulator. What are the first 15 numbers pushed onto the queue by the following program (which runs forever)?

```
WOB  DC    0                  STORE URN
URN  DC    1                  POP
     LOAD  URN                STORE WOB
     PUSH                     ADD   URN
C    MULT  =0                 PUSH
     PUSH                     SUB   URN
I    LOAD  WOB                BE    C
(continued on right)          BU    I
```

Question 19. Compute $f(3,4)$ if

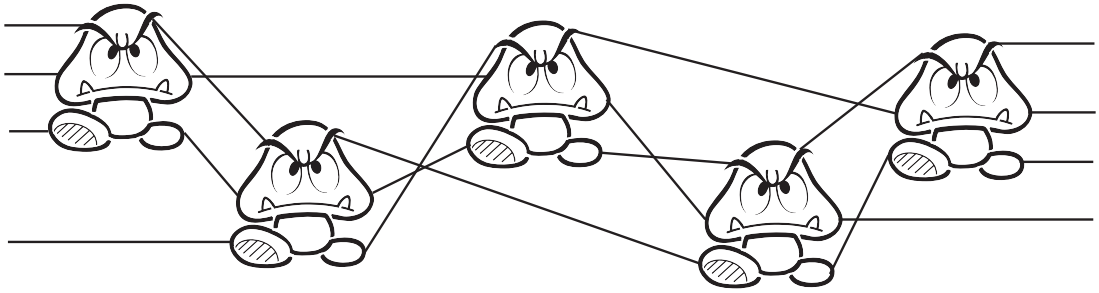
$$f(u,n) := \begin{cases} u+n, & \text{if } u > n; \\ n^2 & \text{if } u = 0; \\ f(u-1,n-1) + f(u+1,n-1) & \text{otherwise.} \end{cases}$$

Question 20. Convert $\frac{(n(n+1))(2n+1)}{6}$ to prefix.

Question 21. The *Fredkin gate* has three input wires (I_1, I_2, I_3) and three output wires (O_1, O_2, O_3) defined by the following properties:

- $O_1 = I_1$
- if I_1 is false then $O_2 = I_2$ and $O_3 = I_3$
- if I_1 is true then $O_2 = I_3$ and $O_3 = I_2$

We represent a Fredkin gate by a goomba, where the inputs and outputs are 1 on top, 2 in the middle, and 3 on the bottom. If the values on the output wires (at the right) of the following circuit are 1001 from top to bottom (where 0 is false, 1 is true), what are the inputs from top to bottom?



Question 22. Solve for X: LCIRC-2 X XOR RSHIFT-2 X AND 10101 = NOT X OR 10101

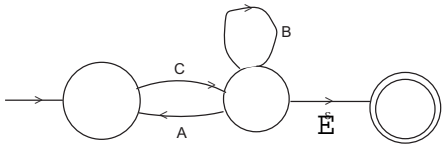
Question 23. A typical way to evaluate postfix expressions is via a stack: read the input expression from left to right, push each number encountered onto the stack, apply each operator encountered to the top two items of the stack and replace them by the result, and at the end there should be a single number on the stack which is your final answer. Suppose we introduce a new type of symbol §, which swaps the top element of the current stack with the bottom element. Compute the values of the following expressions.

$$3\,4\,*\,5\,+$$

$$4\,9\,1\,\S\,+\,*$$

$$9\,1\,3\,2\,5\,4\,\S\,*\,\S\,+\,\S\,*\,\S\,+\,\S\,*$$

Question 24. Write a simplified regular expression for the set of strings accepted by the following finite state automaton.



Question 25. What is output by this program?

```
 2 for J = 1 to 10
 3   X$(J) = ""
 5 next J
 7 read F$
11 for K = 1 to len(F$)-1
13   for M = 1 to 10
17     if (len(X$(M))>0) and (mid$(F$,K,1) <> left$(X$(M),1)) then 27
19     X$(M) = X$(M) + mid$(F$,K,2)
23     goto 31  [typo: line 23 should say 'goto 29']
27   next M
29 next K
31 data BANANARAMA
37 for J = 1 to 10
41   if (len(X$(J)) = 0) then 47
43   print(X$(J))
47 next J
```

Question 26. Consider the following recursive function (the Calkin-Wilf tree):

$$f(n) := \begin{cases} 1, & \text{if } n = 1; \\ 1/(1 + 1/f(n/2)), & \text{if } n \text{ is even}; \\ f((n-1)/2) + 1, & \text{otherwise.} \end{cases}$$

Find $f(12)$.

Bonus Problems

Question 27. ACSL Trivia. What is the overflow behaviour of ACSL assembler?

1. Values 1000000 or greater cause the program to crash.
2. After each operation, the accumulator's value *acc* is set to $acc \mod 1000000$.
3. The accumulator can hold arbitrarily large values.
4. Need more overlords.

Question 28. A gate with the same number of input wires as output wires is *reversible* if you can always uniquely determine the input from the output, and *conservative* if the number of output 1s always equals the number of input 1s. Which of the Toffoli gate (Question 12) and Fredkin gate (Question 21) are reversible? Which are conservative? Are all conservative gates reversible?

Question 29. What does `RESTORE` in Basic do?

Question 30. Draw an FSA which recognizes the language over the alphabet $\{FSA\}$ of all strings in which *each F is immediately followed by at least one S and each SS is immediately followed by at least one A.*

Question 31. For f defined in Question 26: show every positive rational number can be expressed as $f(i)$ for some integer i. Mega-bonus: prove $f(n+1) = 1/(2\lfloor f(n) \rfloor + 1 - f(n))$.