Approximability of Sparse Integer Programs

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Motivation

- Integer linear programming (IP) is a classical NP-complete problem
- Lenstra 1983: for a fixed number of variables, IP is poly-time solvable
  - Also for fixed number of constraints
- What can we say if the number of variables *per constraint* is fixed?
  - Or number of constraints per variable?
- How hard are these problems?

4 Families of Problems

- We consider two common and natural types of IP
  - *Packing* IPs: $\{\max cx \mid Ax \leq b, x \geq 0\}$
  - *Covering* IPs: $\{\min cx \mid Ax \geq b, x \geq 0\}$
- We consider *sparse* IPs (refers to pattern of nonzeroes of constraint matrix A)
  - *k-row sparse* means at most k variables are involved in each constraint
  - *k-column sparse* means each variable is involved in at most k constraints

What are these problems?

in simplest case A & b 0-1, c=1, k=2:

	CS (k occurrences/var)	RS (k vars/constraint)
PIPs		
CIPs		

What are these problems?

in simplest case A & b 0-1, c=1, k=2:

	CS (k occurrences/var)	RS (k vars/constraint)
PIPs	maximum matching (poly-time)	max independent set (NP-complete)
CIPs	minimum edge cover (poly-time)	min vertex cover (NP-complete)

- NP-hard for general A or k=3

Approximability Bounds, Then

	CS (k occurrences/var)	RS (k vars/constraint)
PIPs	$\geq \Omega(k/\ln k)$ [Hazan et al '03] (k-set matching)	$\geq \Omega(\#\ \text{vars}^{1-o(1)})$ [Khot-Ponnuswami '06] (independent set)
CIPs	$\geq \ln k - O(\ln \ln k)$ [Trevisan '01] (k-set cover)	$\geq k-\varepsilon$ [UGC + Khot-Regev '03] (hypergraph vertex cover)

Approximability Bounds, Now

	CS (k occurrences/var)	RS (k vars/constraint)
PIPs	$\geq \Omega(k/\ln k)$ [Hazan et al '03] our paper: $\leq k^2 2^k$ Bansal et al: $\leq O(k)$	$\geq \Omega(\#\text{ vars}^{1-o(1)})$ [Khot-Ponnuswami '06] $\leq \varepsilon\cdot(\#\text{ vars})$ [using Lenstra '83]
CIPs	$\geq \ln k - O(\ln \ln k)$ [Trevisan '01] $\leq O(\ln k)$ [Kolliopoulos-Young '01]	$\geq k-\varepsilon$ (mod UGC) [Khot-Regev '03] our paper: $\leq k$ also Koufogiannakis-Young

k-Column-Sparse Packing

- Our paper uses iterated LP relaxation to get super-optimal solution with additive violation, then try to remove violation
- $O(k^2 2^k)$-approx in paper; same framework improved to $O(k^2)$-approx later (CEK & CP)
- Bansal, Korula, Nagarajan in August:
  - Nice $O(k)$-approx via randomized rounding
  - Works for submodular objective functions

Iterated Rounding Result

- Lemma: for extreme point solution x to the LP {max cx : Ax ≤ b, 1 ≥ x ≥ 0}, either
  - (round) $x_e = 1$ for some e
  - (relax) there is a vertex v* such that at most k edges e have (v* in e) and ($x_e > 0$)
- Iterated rounding then gives solution
  - with value ≥ LP-OPT
  - but violating each constraint by up to +k

k-Row-Sparse Covering

- We get *direct LP-rounding* k-approx alg
- Insight: direct rounding depends on a property of each individual constraint
- The ith constraint is *k-roundable* if

for all real nonnegative x with $\sum_j a_{ij} x_j \geq b_i$,

$$\sum_j a_{ij} \mathrm{floor}(k x_j) \geq b_i$$

- If all k-roundable, $\mathrm{floor}(k x^{OPT})$ is a k-apx
  - x^{OPT}:= optimal solution to LP relaxation of IP

Getting k-Roundability

- k-RS constraint has ≤ k variables
- Not all k-RS constraints are k-roundable unfortunately
- Define two constraints to be *equivalent* if they have same solutions in Z_+
- Main lemma: every k-RS constraint is *equivalent* to a k-roundable constraint

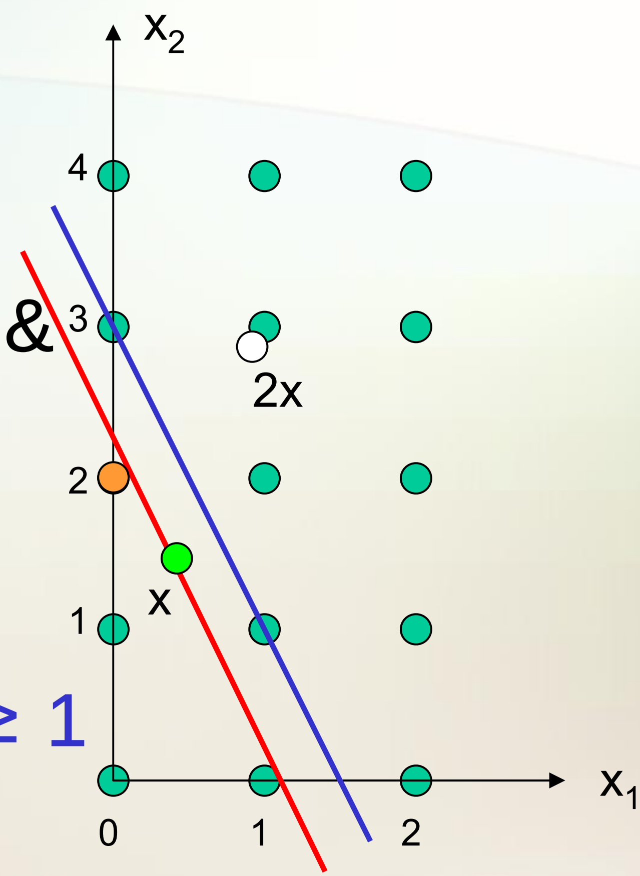
Illustration of Lemma

- Example with k=2

$0.99x_1 + 0.49x_2 \geq 1$ is not 2-roundable.

Proof: $x=(0.4, 1.4)$ feasible &

floor(2x)=(0,2) not feasible



However the constraint is <u>equivalent</u> to ${}^{2}/_{3}x_1 + {}^{1}/_{3}x_3 \geq 1$ which ***is*** 2-roundable

Lemma Proof Idea

- Scale constraint $\sum_j a_{ij} x_j \geq b_i$ so that $b_i = 1$
- Cap each a_{ij} at 1
  - This preserves integer feasible set
- Short calculation shows this constraint is ρ-roundable for $\rho = 1 + \sum_j a_{ij}$
- Thus we are good if $\sum_j a_{ij} \leq k-1$
- Some easy ad-hoc case analysis takes care of case that $\sum_j a_{ij} > k-1$

Overall k-RS CIP Algorithm

- Replace each row with an equivalent *k*-roundable one
- Solve the LP to get optimal solution x*
- Output floor(kx*)
  - Can also handle multiplicity constraints x ≤ d with *knapsack cover inequalities*
  - Koufogiannakis-Young's approach: Simple fast iterated primal alg, works for broad generalization, no integrality gap bound

Future Work

- Remaining open problems for k-CS PIPs:
  - Find O(k) approximation without solving LP
  - Close gap of approximability between Ω(k/ln k) and O(k)
    - even in 0-1 case, i.e. k-set packing
- Generalizations of sparse IPs
  - Semimodular objective
  - Monotone sparse constraints

Thanks for your attention!