Algorithms & LPs for k-Edge Connected Spanning Subgraphs

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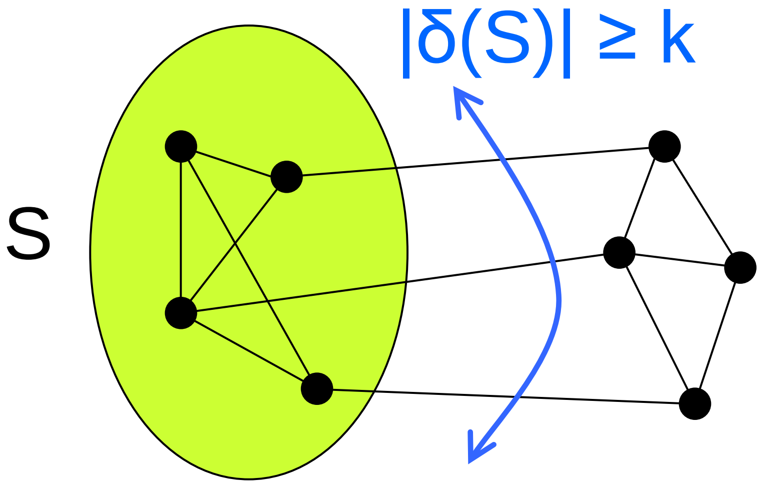
k-Edge Connected Graph

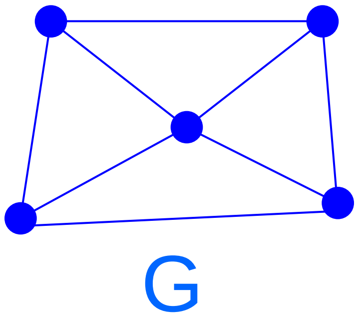
- k edge-disjoint paths between every u, v
- at least k edges leave S, for all $\emptyset \neq S \subsetneq V$
- even if (k-1) edges fail, G is still connected

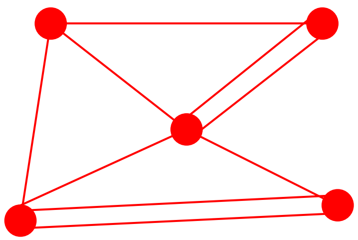


k-ECSS & k-ECSM Optimization Problems

k-edge connected spanning subgraph problem (k-ECSS): given an initial graph (maybe with edge costs), find k-edge connected subgraph including all vertices, w/ |E| (or cost) minimal

k-ecs multisubgraph problem (k-ECSM): can buy as many copies as you like of any edge





3-edge-connected multisubgraph of G, |E|=9

Question

What is the approximability of min-cost k-edge-connected spanning subgraph and k-edge-connected spanning multisubgraph?

- How does it depend on k?
- Also look at important *unit-cost* special case:

[GGTW05] for unit-cost k-ECSS
$\forall k$, ratio $1+{}^{2}/_{k}$ is possible by **LP methods**;
$\forall k>1$, ratio $1+{}^{0.0001}/_{k}$ impossible unless P=NP

Menu

Appetizer

Conjecture: k-ECSM with general costs can be apx within 1+O(1/k) & doable via LP

integrality gap 1+O(1/k)

Entrée

For k-ECSS with general costs, we prove:
∀k, ratio 1.003 is not possible unless P=NP

Dessert

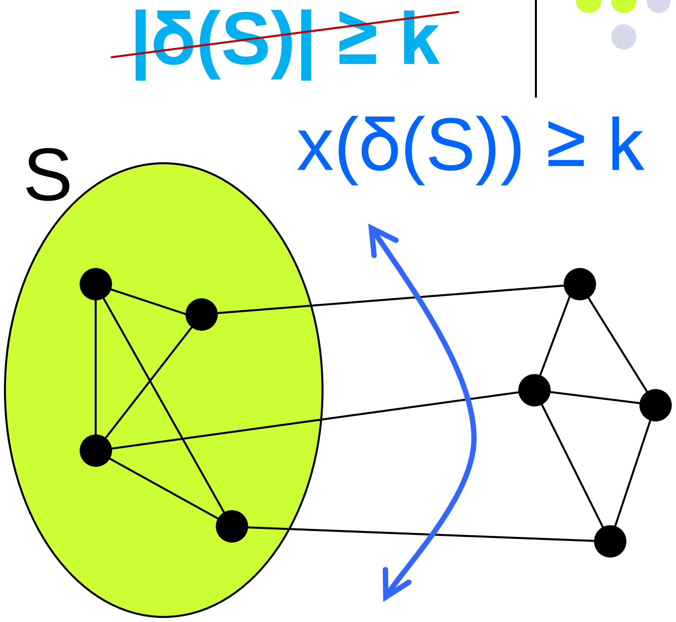
Discovered new complexities of LP relaxation

Appetizer

- **Conjecture**: k-ECSM admits approx. ratio 1+O(1/k), and same integrality gap
- Bang-Jensen & Yeo '01 "Splitting Conjecture"
  - Is there a constant C such that $\forall t$, every $(2t+C)$-edge-connected graph can be decomposed into two edge-disjoint t-edge-connected subgraphs?
- We prove that if the answer is yes the integrality gap is indeed at most 1 + C/k

Proof Ideas (1/2)

- LP:
  - variable $x_e \geq 0$ for each edge e
  - for every nonempty $S \subsetneq V$, $x(\delta(S)) \geq k$



- Take a feasible x and scale it up by a factor μ to become integral, we have a kμ-edge-connected graph;
  - or scale up by μt ⇒ kμt-edge-connected

Proof Ideas (2/2) – Splitting

Splitting Conj. ∀t, every (2t+C)-edge-connected graph contains 2 edge-disjoint t-edge-connected subgraphs

(4t+3C)-con

(2t+C)-con

(2t+C)-con

t-con

t-con

t-con

t-con

Implies ∀t ∀x, any (2^xt+(2^x-1)C))-edge-connected graph contains 2^x edge-disjoint t-edge-connected subgraphs

Another Intriguing Question

- Company has a k-edge-connected network
- Want to sell a spanning tree and retain as much edge-connectivity as possible
- How much edge-connectivity can we keep by a judicious choice of tree to sell? **=: r(k)**

Nash-Williams/Tutte

Best known bounds: k-3 ≥ r(k) ≥ floor(k/2)-1

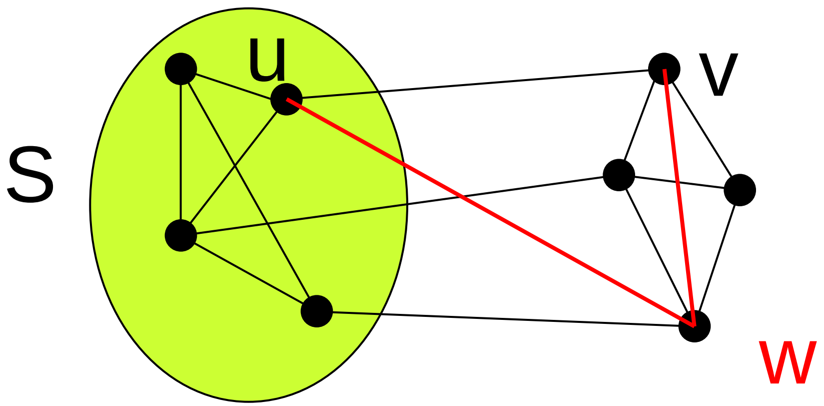
Splitting Conjecture implies r(k) ≥ k - O(log k)

Entrée
Approximation Hardness

For the k-ECSM (multisubgraph) problem, we may assume edge costs are metric, i.e.

$$\mathrm{cost}(uv) \le \mathrm{cost}(uw) + \mathrm{cost}(wv)$$

since replacing uv with uw, wv maintains k-EC

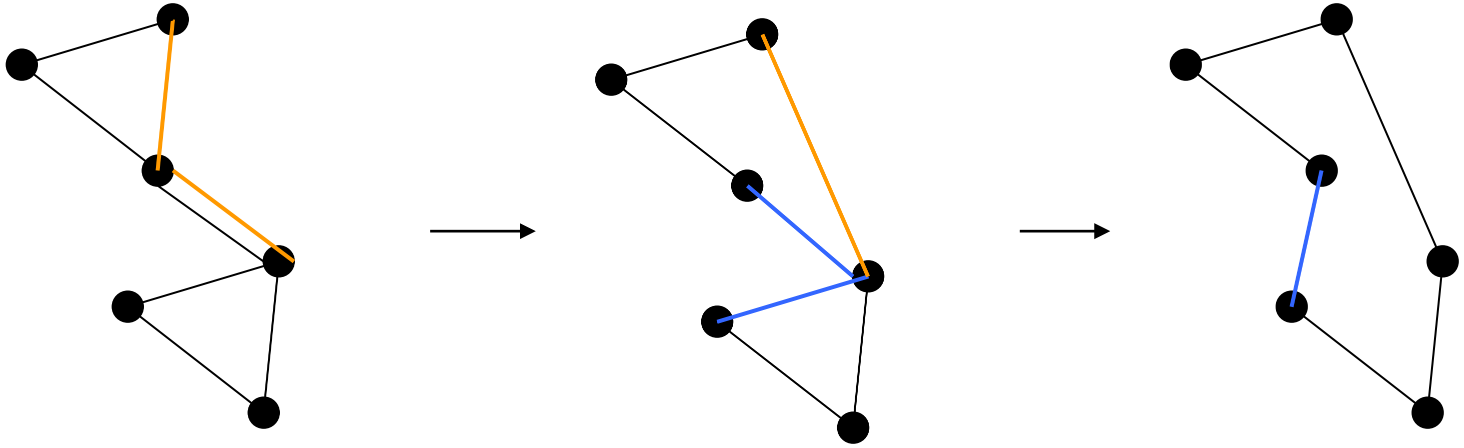


What's Hard About Hardness?

A 2-VCSS is a 2-ECSS is a 2-ECSM.

vertex-connected

For metric costs, can *split-off* conversely, e.g.

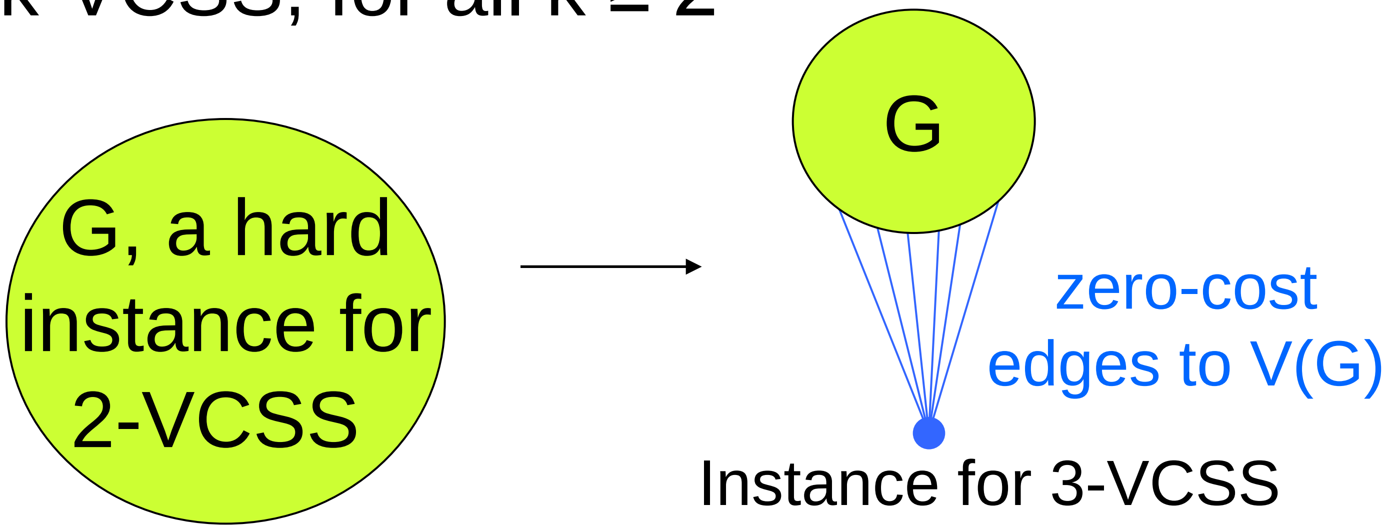


2-ECSM 2-ECSS 2-VCSS

All of these are APX-hard [via {1,2}-TSP]

What's Hard About Hardness?

1+ε hardness for 2-VCSS implies 1+ε hardness for k-VCSS, for all k ≥ 2



Instance for 3-VCSS with same hardness

But this approach fails for k-ECSS, k-ECSM

Hardness of k-ECSS (slide 1/2)

∃ ε>0, ∀ k≥2, no 1+ε-apx if P ≠ NP

Reduce APX-hard TreeCoverByPaths to k-ECSS

Input: a tree T, collection X of paths in T

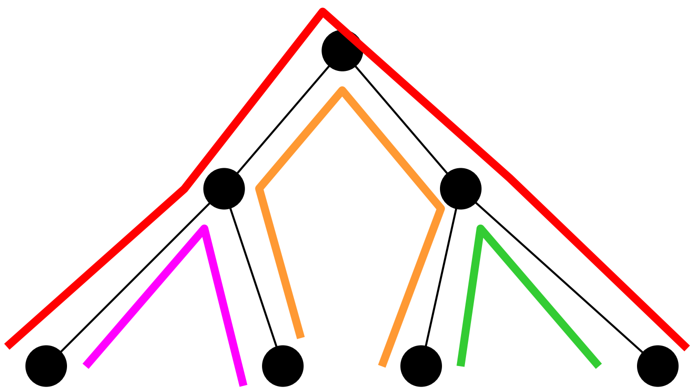
A subcollection Y of X is a *cover* if the union of {E(p) | p in Y} equals E(T)

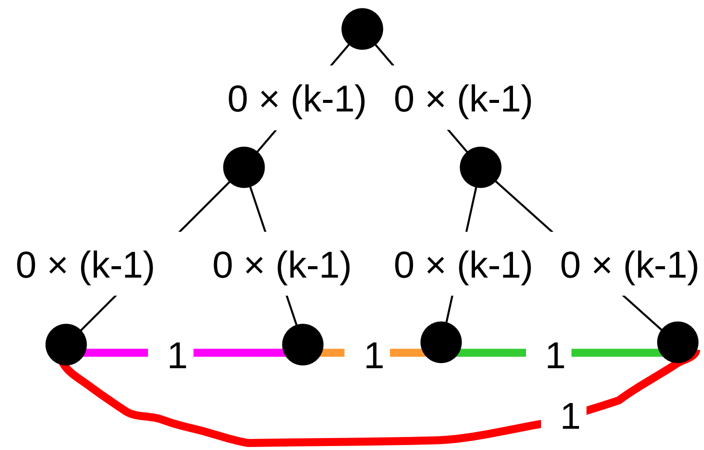
Goal: min-size subcollection of X that is a cover

Hardness of k-ECSS (slide 2/2)

$\exists\ \varepsilon>0,\ \forall\ k\geq 2$, no $1+\varepsilon$-apx if P $\neq$ NP

- Replace each edge e of T by k-1 zero-cost parallel edges; replace each path p in X by a unit-cost edge connecting endpoints of p



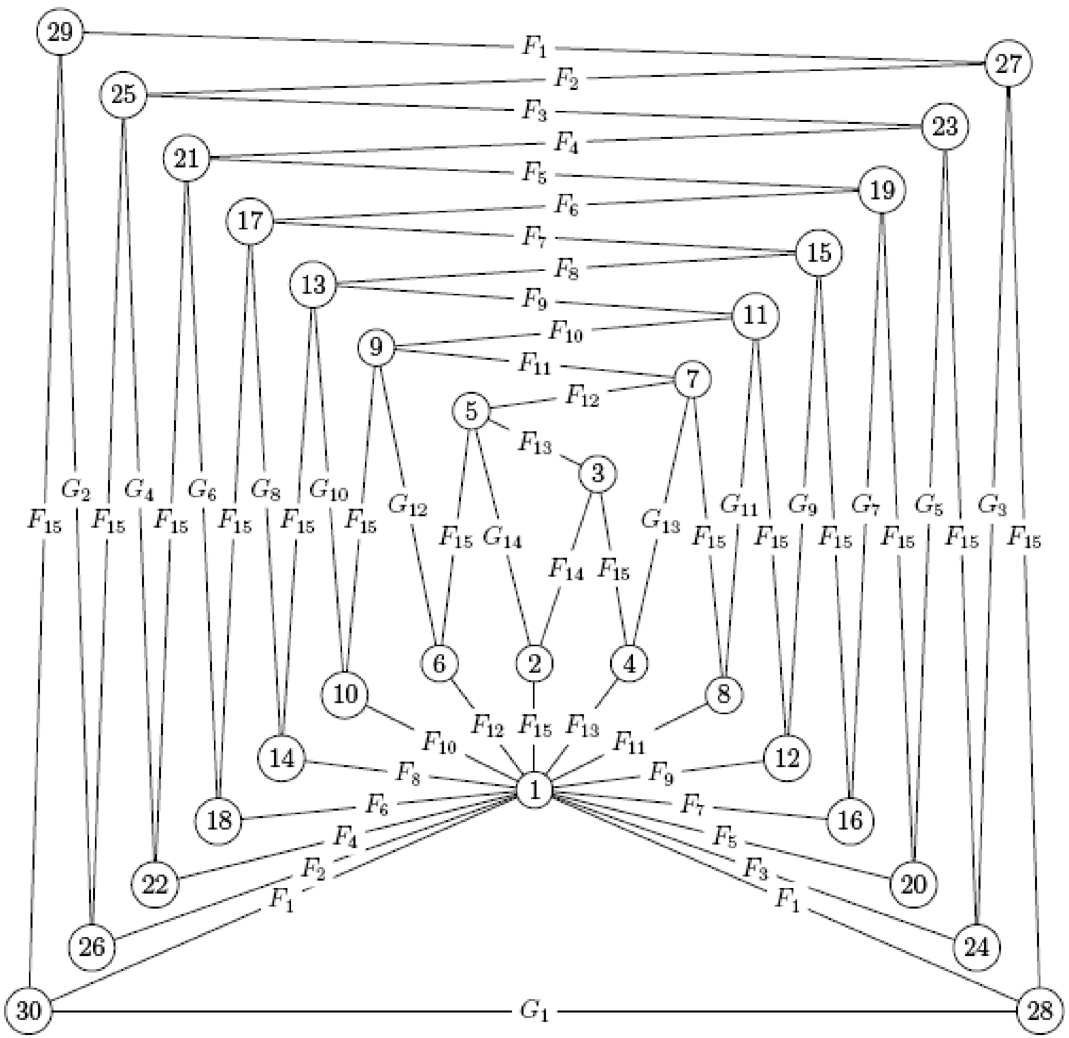


… min |X| to cover T = k-ECSS optimum.

Dessert
Extreme Points of the LP

- LP [x(δ(S)) ≥ k] is a scaled version of:
  - Held-Karp relaxation of TSP
  - Undirected cut relaxation of Steiner tree
- Has "Parsimonious Property" [GB93]
  - LP-based approx. algorithm for k-ECSM gives "for free" an algorithm for *subset k-ECSM*
- Nice structural properties are key for LP-based algorithms (e.g.GGTW). What ugliness exists?

Extremely Extreme Extreme Point



- Edge values of the form $Fib_i/Fib_{|V|/2}$ and $1 - Fib_i/Fib_{|V|/2}$ (exponentially small in |V|)
- Maximum degree |V|/2

Digestif
One is open, one is false

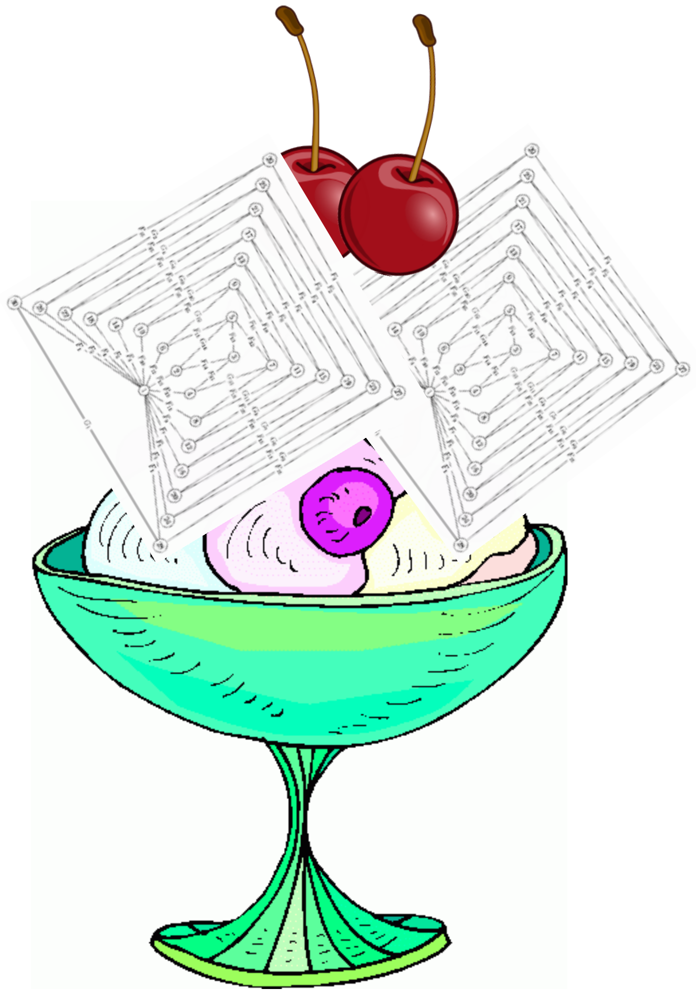
∃k, each k-strongly edge-connected digraph has 2 disjoint strongly connected subdigraphs

OPEN [B-J Y 01]

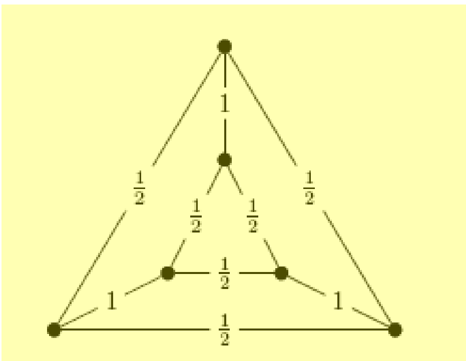
∃k, every k-edge-connected hypergraph has 2 disjoint connected hypergraphs

FALSE [B-J T 03]

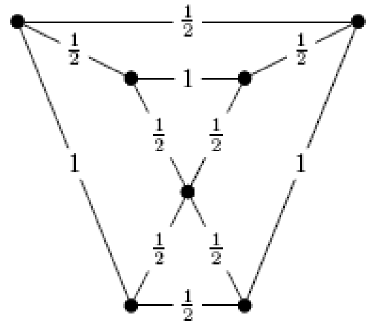
Thanks for Attending!



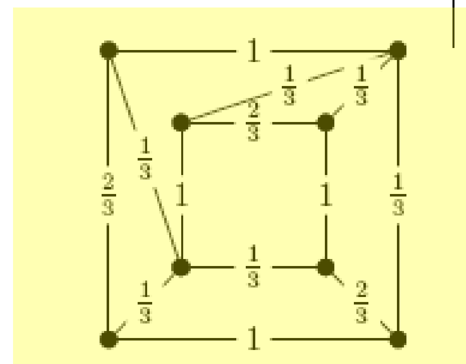
Small Extreme Examples



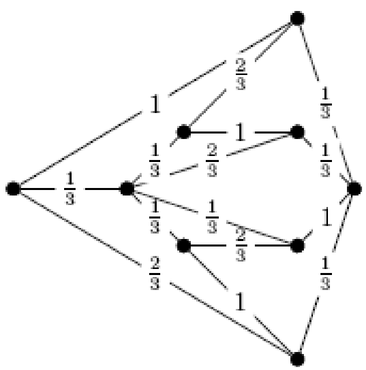
n=6, denom=2



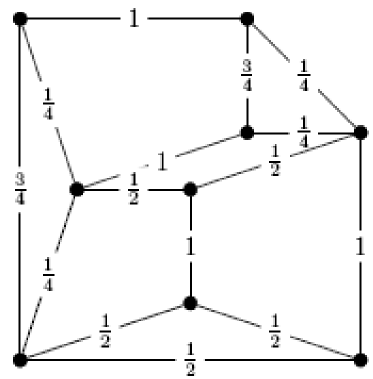
n=7, Δ=4



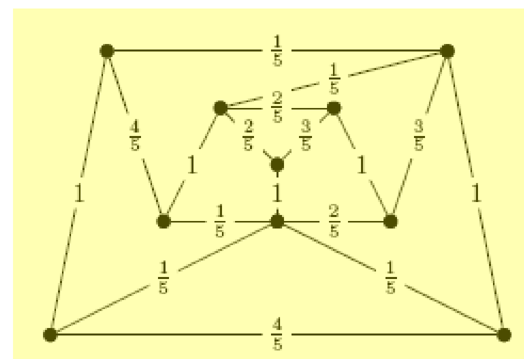
n=8, denom=3



n=9, Δ=5

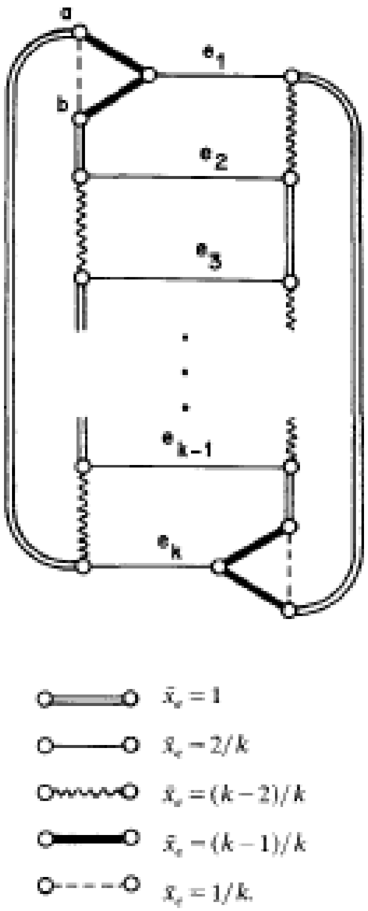


n=9, denom=4



n=10, denom=Δ=5

Previously Known Constructions



[BP]: minimum nonzero value of x* can be ~1/|V|

[C]: max degree can be ~$|V|^{1/2}$