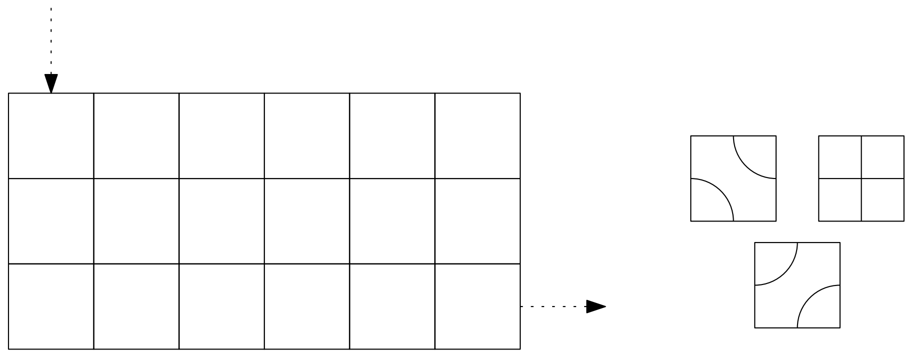
Every two-player finite game has either a winning strategy for the first player, or the second player. For the games below, the winner depends on the board size or shape. Can you determine the winner in all situations?

Trailblazer



Board: an $M \times N$ board and an infinite supply of the three pieces shown.
Moves: place a piece in an empty square, in such a way that the trailhead is extended.
Winning: you win if you make the trail reach the exit; you lose if you make the trail go out of bounds.

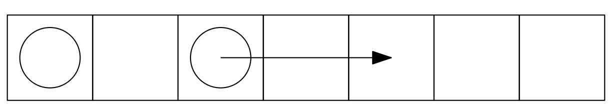
Nim

Board: several piles of counters.
Moves: choose a single pile, and take away any number of counters from that pile. Empty piles are deleted.
Winning: you win if you take the last counter.

Moore's Nim

Board: several piles of counters.
Moves: choose a single pile, and take away any number of counters from that pile. Optionally, choose a second pile and remove one or more counters from it. Empty piles are deleted.
Winning: you win if you take the last counter.
Note: Like Nim, but not binary.

Nim Slide



Board: a row of cells, some containing counters and some empty.
Moves: take any counter and slide it one or more cells to the right; you cannot slide two counters into the same cell, and you cannot slide one counter over/past another.
Winning: the last person to make a move wins. (At the end the counters are in consecutive right-end cells.)
Note: for starters, analyze this game when where are only a few counters (e.g. 3 or 4).

Treblecross

Board: a row of N cells, initially empty.
Moves: put an X in any empty cell.
Winning: whoever causes three Xs in a row wins.
Note: requires Sprague-Grundy theory, which is based on Nim. You will not get a general formula in terms of N, but you will be able to easily compute who wins for a given value of N in an efficient way.

Note: the remaining problems don't require Nim.

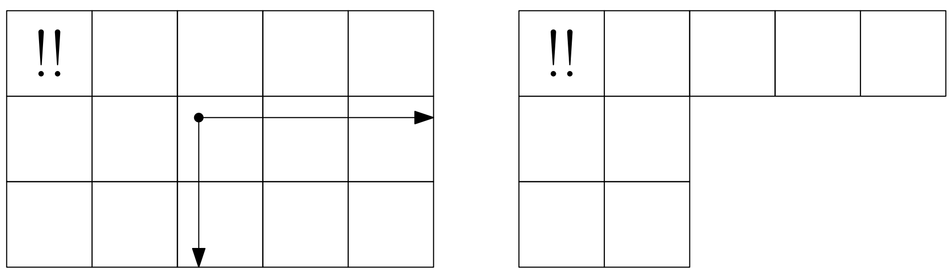
Toonie Table

Board: a rectangular table, originally empty.
Moves: put a toonie on the table, so that it doesn't overlap any of the previous ones.
Winning: the first person unable to move loses.

SOS (1999 USAMO)

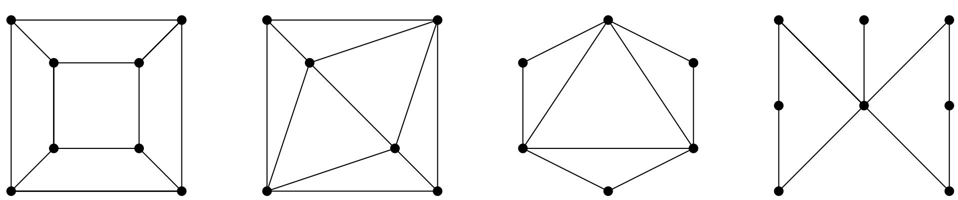
Board: an empty 1×2000 grid.
Moves: write an S or an O in an empty cell.
Winning: whoever causes SOS to be spelled. If the board fills up, it is a draw.
Show that the second player has a winning strategy.

Chomp



Board: an $M \times N$ bar of chocolate. The top-left corner (!!) is poisoned.
Moves: pick any uneaten cell and *chomp* it, which deletes it and all cells below/to the right of it.
Losing: you lose if you eat the poison!
Note: the first player always has a winning strategy. Prove it by using a non-constructive *strategy stealing* argument: assume that the second player has a winning strategy, and then use it to construct a winning strategy for the first player.
Note 2: an explicit description of a general winning strategy is not known.

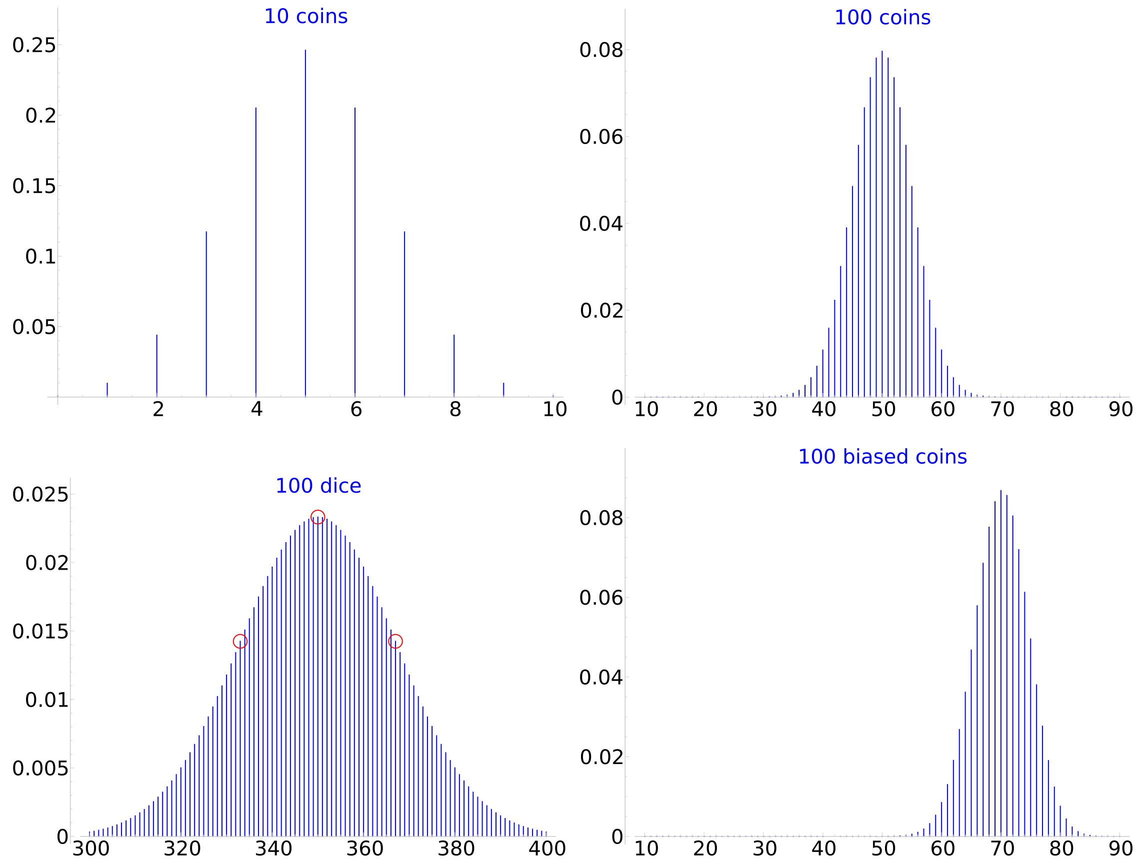
Snakes on a Graph



Board: a graph (drawing) with vertices (points) and edges (line segments). Two vertices are *adjacent* if they are connected by an edge. Four sample graphs are given above.
Moves: the first player puts a snake on any vertex. On every later turn, the player moves the snake to any vertex that (1) is adjacent to its current position AND (2) has never been visited before.
Winning: the first player unable to move loses.
Note: Can you give a very simple strategy for player 2 to win on two of the graphs above? It is a little more difficult, and requires some graph theory, to give an optimal general strategy for player 1 on the other two graphs.

Probability and Games

The top left diagram shows, for each value $0 \le k \le 10$, the probability when we flip k coins, the probability that we get exactly k heads. Top right: same with 100 coins. Bottom left: same with 100 dice (measuring what's the sum of all 100 numbers on top); here the inflection points and maximum are circled. Bottom right: 100 *biased* (unfair) coins, which each have probability 0.7 of having heads on top.



The *central limit theorem* says that any random variable, when added to itself enough times, approaches this same shape (a *normal distribution*). It is centred around the *expected value*, which grows linearly with the number N of trials. (The *standard deviation* is the distance from centre to inflection point, and grows as fast as the square root of N.)

Expected Value

Let X be a random variable. So X can take on some possible values $a, b, \ldots, z$ where $\Pr[X = a] + \cdots + \Pr[X = z] = 1$ and all of these probabilities are non-negative. Then its *expected value*, denoted $\mathrm{E}[X]$, is the weighted average of the values it can take on, where the weights correspond to the probabilities:

$$\mathrm{E}[X] = \sum_{\text{all values } v \text{ that } X \text{ could be}} v \times \Pr[X = v].$$

For example if X can only take on two values, a with probability $1/3$ and b with probability $2/3$, then $\mathrm{E}[X] = a \times 1/3 + b \times 2/3$.

Verify that the expected value of the number on top of a die is 3.5. Then note, the bottom-left diagram, which has $N = 100$, is centred around $3.5 \times 100 = 350$; the central limit theorem guarantees this in the limit. A nice exercise is proving more generally that $\mathrm{E}[A + B] = \mathrm{E}[A] + \mathrm{E}[B]$ for any random variables A and B.

The Matching Game

This is a 2-player zero-sum game where both players pick an integer between 1 and 10. If the numbers match, then player 1 wins, otherwise player 2 wins. The loser pays the winner \$1. What is an optimal strategy for this game? Each round, what is the expected profit of each player?

The game where the coins are actually worth money

Each round, both players pick our a coin worth \$1, \$2, or \$3 from their wallet. If they match then player 1 takes both, otherwise player 2 takes both. What are the optimal strategies?

Quadratic mean $\geq$ Arithmetic mean

Show that $\mathrm{E}[(X - \mathrm{E}[X])^2] = \mathrm{E}[X^2] - (\mathrm{E}[X])^2$. Using this prove $(a^2 + b^2 + c^2)/3 \geq ((a + b + c)/3)^2$.

Crown and Anchor

The University of Waterloo is raising money by starting a casino. They run *Crown and Anchor*, a traditional game with 6 symbols. Every time the game runs, three dice with these symbols on the sides are rolled. To gamble, you bet \$1 on a symbol before the roll takes place. If your symbol shows up on $T > 0$ dice, then you get your \$1 back plus \$T extra. If your symbol does not show up on any dice, you lose the \$1 that you bet. Will UW go bankrupt?

Inclusion-Exclusion ($|A \cup B| = |A| + |B| - |A \cap B|$)

- Pick a random integer from 1 to 200. What is the probability that it does not contain the digit 0?
- Juan rolls a fair regular eight-sided die. Then Amal rolls a fair regular six-sided die. What is the probability that the product of the two rolls is a multiple of 3? (2002 AMC 12B 16)
- Find a formula for $|A \cup B \cup C|$ in terms of $|A|, |B|, |C|, |A \cap B|, |B \cap C|, |A \cap C|, |A \cap B \cap C|$. Find also a formula for $|A \cap B \cap C|$ in terms of $|A|, |B|, |C|, |A \cup B|, |B \cup C|, |A \cup C|, |A \cup B \cup C|$.

Random Problems

- You pick two random integers from 1 to 8, independently and with replacement. (Like rolling two 8-sided dice.) What is the probability that the numbers' product is greater than their sum?
- You roll three six-sided dice. What is the probability that the three numbers on top can be used to form the sides of a non-degenerate triangle?
- A box contains exactly five chips, three red and two white. Chips are randomly removed one at a time without replacement until all the red chips are drawn or all the white chips are drawn. What is the probability that the last chip drawn is white? (2001 AMC 12 11)
- A bag contains 2 red onions and 1 white onion. Every minute for one hour, you pull a random onion out of the bag, and then put it back in, along with an extra onion of the same colour. After one hour there are 61 onions in the bag. You pick one out a random. What is the probability that it is red?

Challenge: Gambler's Ruin

You enter the casino with \$30. The game you love to play is simple: flip a coin, and if it is heads you win \$1, and if it is tails, you lose \$1. You decide to keep playing this game over and over until either (i) you run out of money, or (ii) your total rises to \$100. What is the probability of (i), and the probability of (ii)?
Hint: There is a beautiful proof based on evaluating your expected value as a function of time.

Coin Manufacturing

You are trapped on an desert island, gambling with your friend. You only have one coin, and it is biased. You don't even know what probability it has to come up heads. How can you use this biased coin to simulate a fair coin?

Simpson's Paradox

In January, Homer won 1 out of 5 league StarCraft matches. In February, he won 3 out of 4 league matches. His wife Marge won 2 out of 9 matches in January and 4 out of 5 in February. Who was the better player each month? Who was better overall?

Probabyility

The country of Probabilia has a population with more men than women. The president has declared that a new policy will be started in order to increase the number of female births:

Policy: When a family has a son, they are not allowed to have any more children.

The President thinks this is a good idea because then you can have some families with several daughters, but every family will have at most one son. Prove that it will not actually have any effect on the overall ratio of daughters born to sons born. On the other hand, prove that if you consider the percent of children in every family that are boys, and take the average percent over all families, that this will be *greater* than 1/2.

Your Dilemma

You were arrested for being a robber. But the prince has made you an offer. He gave you two bags, 10 skull coins, and 10 happy face coins. First, you get to arrange the coins in the bags however you like; each coin must be in a bag. Second, a random bag will be selected. Third, a random coin from that bag will be selected. If it is a skull you go to jail, but if it is a happy face you go free. How can you minimize your probability of going to jail? (Hint: it is much less than 1/2.)

Let's Make a Deal

You are on a game show where there are three shiny boxes. The host, Guy Smiley, shows you that one box contains a gold coin, and the two other boxes contain turnips. (Assume you do not want a turnip, even though they are very nutritious.) The boxes are then closed and shuffled randomly. The rules say

first that you pick one box, without opening it. Guy Smiley will then open a different box, and in particular he always opens a box containing a turnip, and eats it. That leaves two boxes: the one you picked, and another one you didn't pick. Finally, you have the choice to keep your originally chosen box, or switch to the other unopened box. If the one you end up with has the gold coin you keep it!

Is it better to keep your original box, or switch boxes? Hint: even though you have 2 choices and you don't know which one has the turnip, it is not true that both boxes are 50% likely to contain the gold coin.

The Minimum Unique Integer Game

You can play the following game with any number of people. Each one picks a positive integer, and everyone's choice is revealed at the same time. The person who chose the minimum *unique* integer wins. For example if 4 people are playing and the choices are 1, 4, 1, 3, then the winner is the person who chose 3 (the 1s are not unique, and 3 is the smallest remaining integer). It is possible that nobody wins, for example if the choices are 2, 2, 3, 3. Play this game with your friends! Show that for two players it is not very interesting.
Note: for three players, you can show that any *symmetric equilibrium* has to use all positive integers with nonzero probability (even very large ones like 1000).